THE
THEORY OF SOUND.

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AERIAL VIBRATIONS.

236. SINCE the atmosphere is the almost universal vehicle of sound, the investigation of the vibrations of a gaseous medium has always been considered the peculiar problem of Physical Acoustics; but in all, except a few specially simple questions, chiefly relating to the propagation of sound in one dimension, the mathematical difficulties are such that progress has been very slow. Even when a theoretical result is obtained, it often happens that it cannot be submitted to the test of experiment in default of accurate methods of measuring the intensity of vibrations. In some parts of the subject all that we can do is to solve those problems whose mathematical conditions are sufficiently simple to admit of solution, and to trust to them and to general principles not to leave us quite in the dark with respect to other questions in which we may be interested.

In the present chapter we shall regard fluids as perfect, that is to say, we shall assume that the mutual action between any two portions separated by an ideal surface is *normal to that surface.* Hereafter we shall say something about fluid friction; but, in general, acoustical phenomena are not materially disturbed by such deviation from perfect fluidity as exists in the case of air and other gases.

The equality of pressure in all directions about a given point is a necessary consequence of perfect fluidity, whether there be rest or motion, as is proved by considering the equilibrium of a small tetrahedron under the operation of the fluid pressures, the

impressed forces, and the reactions against acceleration. In the limit, when the tetrahedron is taken indefinitely small, the fluid pressures on its sides become paramount, and equilibrum requires that their whole magnitudes be proportional to the areas of the faces over which they act. The pressure at the point x, y, z will be denoted by p.

237. If $\rho X dV$, $\rho Y dV$, $\rho Z dV$, denote the impressed forces acting on the element of mass ρdV, the equation of equilibrium is

$$dp = \rho\,(X dx + Y dy + Z dz),$$

where dp denotes the variation of pressure corresponding to changes dx, dy, dz in the co-ordinates of the point at which the pressure is estimated. This equation is readily established by considering the equilibrium of a small cylinder with flat ends, the projections of whose axis on those of co-ordinates are respectively dx, dy, dz. To obtain the equations of motion we have, in accordance with D'Alembert's Principle, merely to replace X, &c. by $X - \frac{Du}{Dt}$, &c., where $\frac{Du}{Dt}$, &c. denote the accelerations of the particle of fluid considered. Thus

$$\left.\begin{aligned} \frac{dp}{dx} &= \rho\left(X - \frac{Du}{Dt}\right) \\ \frac{dp}{dy} &= \rho\left(Y - \frac{Dv}{Dt}\right) \\ \frac{dp}{dz} &= \rho\left(Z - \frac{Dw}{Dt}\right) \end{aligned}\right\} \quad \text{..........................(1).}$$

In hydrodynamical investigations it is usual to express the velocities of the fluid u, v, w in terms of x, y, z and t. They then denote the velocities of the particle, whichever it may be, that at the time t is found at the point x, y, z. After a small interval of time dt, a new particle has reached x, y, z; $\frac{du}{dt}\,dt$ expresses the excess of its velocity over that of the first particle, while $\frac{Du}{Dt}\,dt$ on the other hand expresses the change in the velocity of the *original* particle in the same time, or the change of velocity at a point, which is not fixed in space, but moves with the fluid. To this notation we shall adhere. In the change contemplated in $\frac{d}{dt}$, the

position in space (determined by the values of x, y, z) is retained invariable, while in $\frac{D}{Dt}$ it is a certain particle of the fluid on which attention is fixed. The relation between the two kinds of differentiation with respect to time is expressed by

$$\frac{D}{Dt}=\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\;\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and must be clearly conceived, though in a large class of important problems with which we shall be occupied in the sequel, the distinction practically disappears. Whenever the motion is very small, the terms $u\frac{d}{dx}$, &c. diminish in relative importance, and ultimately $\frac{D}{Dt}=\frac{d}{dt}$.

238. We have further to express the condition that there is no creation or annihilation of matter in the interior of the fluid. If α, β, γ be the edges of a small rectangular parallelepiped parallel to the axes of co-ordinates, the quantity of matter which passes out of the included space in time dt in excess of that which enters is

$$\left\{\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}\right\}\alpha\beta\gamma\,dt;$$

and this must be equal to the actual loss sustained, or

$$-\frac{d\rho}{dt}\alpha\beta\gamma\,dt.$$

Hence

$$\frac{d\rho}{dt}+\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}=0\;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

the so-called equation of continuity. When ρ is constant (with respect to both time and space), the equation assumes the simple form

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

In problems connected with sound, the velocities and the variation of density are usually treated as small quantities. Putting $\rho=\rho_0(1+s)$, where s, called the *condensation*, is small, and neglecting the products $u\frac{ds}{dx}$, &c., we find

$$\frac{ds}{dt}+\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0\;\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

In special cases these equations take even simpler forms. In the case of an incompressible fluid whose motion is entirely parallel to the plane of xy,

$$\frac{du}{dx}+\frac{dv}{dy}=0 \quad\text{..........................(4)},$$

from which we infer that the expression $udy-vdx$ is a perfect differential. Calling it $d\psi$, we have as the equivalent of (4)

$$u=\frac{d\psi}{dy}, \qquad v=-\frac{d\psi}{dx} \quad\text{......................(5)},$$

where ψ is a function of the co-ordinates which so far is perfectly arbitrary. The function ψ is called the *stream*-function, since the motion of the fluid is everywhere in the direction of the curves $\psi=$ constant. When the motion is steady, that is, always the same at the same point of space, the curves $\psi=$ constant mark out a system of pipes or channels in which the fluid may be supposed to flow. Analytically, the substitution of *one* function ψ for the *two* functions u and v is often a step of great consequence.

Another case of importance is when there is symmetry round an axis, for example, that of x. Everything is then expressible in terms of x and r, where $r=\sqrt{y^2+z^2}$, and the motion takes place in planes passing through the axis of symmetry. If the velocities respectively parallel and perpendicular to the axis of symmetry be u and q, the equation of continuity is

$$\frac{d\,(ru)}{dx}+\frac{d\,(rq)}{dr}=0 \quad\text{......................(6)},$$

which, as before, is equivalent to

$$ru=\frac{d\psi}{dr}, \qquad rq=-\frac{d\psi}{dx} \quad\text{....................(7)},$$

ψ being the stream-function.

239. In almost all the cases with which we shall have to deal, the hydrodynamical equations undergo a remarkable simplification in virtue of a proposition first enunciated by Lagrange. If for any part of a fluid mass $udx+vdy+wdz$ be at one moment a perfect differential $d\phi$, it will remain so for all subsequent time. In particular, if a fluid be originally at rest, and be then

set in motion by conservative forces and pressures transmitted from the exterior, the quantities

$$\frac{dv}{dz}-\frac{dw}{dy},\quad \frac{dw}{dx}-\frac{du}{dz},\quad \frac{du}{dy}-\frac{dv}{dx},$$

(which we shall denote by ξ, η, ζ) can never depart from zero.

We assume that ρ is a function of p, and we shall write for brevity

$$\varpi = \int \frac{dp}{\rho} \quad \text{.........................(1).}$$

The equations of motion obtained from (1), (2), § 237, are

$$\frac{d\varpi}{dx} = X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz} \quad \text{.........(2),}$$

with two others of the same form relating to y and z. By hypothesis,

$$\frac{dX}{dy} = \frac{dY}{dx};$$

so that by differentiating the first of the above equations with respect to y and the second with respect to x, and subtracting, we eliminate ϖ and the impressed forces, obtaining equations which may be put into the form

$$\frac{D\zeta}{Dt} = \frac{du}{dz}\xi + \frac{dv}{dz}\eta - \left(\frac{du}{dx} + \frac{dv}{dy}\right)\zeta \quad \text{..............(3),}$$

with two others of the same form giving $\dfrac{D\xi}{Dt}$, $\dfrac{D\eta}{Dt}$.

In the case of an incompressible fluid, we may substitute for $\dfrac{du}{dx}+\dfrac{dv}{dy}$ its equivalent $-\dfrac{dw}{dz}$, and thus obtain

$$\frac{D\zeta}{Dt} = \frac{du}{dz}\xi + \frac{dv}{dz}\eta + \frac{dw}{dz}\zeta, \ \ \text{\&c.} \quad \text{...............(4),}$$

which are the equations used by Helmholtz as the foundation of his theorems respecting vortices.

If the motion be continuous, the coefficients of ξ, η, ξ in the above equations are all finite. Let L denote their greatest numerical value, and Ω the sum of the numerical values of ξ, η, ζ. By hypothesis, Ω is initially zero; the question is whether in

the course of time it can become finite. The preceding equations shew that it cannot; for its rate of increase for a given particle is at any time less than $3L\Omega$, all the quantities concerned being positive. Now even if its rate of increase were as great as $3L\Omega$, Ω would never become finite, as appears from the solution of the equation

$$\frac{D\Omega}{Dt} = 3L\Omega \text{ ....(5).}$$

A fortiori in the actual case, Ω cannot depart from zero, and the same must be true of ξ, η, ζ.

It is worth notice that this conclusion would not be disturbed by the presence of frictional forces acting on each particle proportional to its velocity, as may be seen by substituting $X - \kappa u$, $Y - \kappa v$, $Z - \kappa w$, for X, Y, Z in (1)[1]. But it is otherwise with the frictional forces which actually exist in fluids, and are dependent on the *relative* velocities of their parts.

The first satisfactory demonstration of the important proposition now under discussion was given by Cauchy; but that sketched above is due to Stokes[2]. It is not sufficient merely to shew that if, and whenever, ξ, η, ζ vanish, their differential coefficients $\frac{D\xi}{Dt}$, &c. vanish also, though this is a point that is often overlooked. When a body falls from rest under the action of gravity, $\dot{s} \propto s^{\frac{1}{2}}$; but it does not follow that s never becomes finite. To justify that conclusion it would be necessary to prove that s vanishes in the limit, not merely to the first order, but to all orders of the small quantity t; which, of course, cannot be done in the case of a falling body. If, however, the equation had been $\dot{s} \propto s$, all the differential coefficients of s with respect to t would vanish with t, if s did so, and then it might be inferred legitimately that s could never vary from zero.

By a theorem due to Stokes, the moments of momentum about the axes of co-ordinates of any infinitesimal spherical portion of fluid are equal to ξ, η, ζ, multiplied by the moment of inertia of the mass; and thus these quantities may be regarded

[1] By introducing such forces and neglecting the terms dependent on inertia, we should obtain equations applicable to the motion of electricity through uniform conductors.

[2] *Cambridge Trans.* Vol. VIII. p. 307. B. A. Report on Hydrodynamics, 1847.

as the component rotatory velocities of the fluid at the point to which they refer.

If ξ, η, ζ vanish throughout a space occupied by moving fluid, any small spherical portion of the fluid if suddenly solidified would retain only a motion of translation. A proof of this proposition in a generalised form will be given a little later. Lagrange's theorem thus consists in the assertion that particles of fluid at any time destitute of rotation can never acquire it.

240. A somewhat different mode of investigation has been adopted by Thomson, which affords a highly instructive view of the whole subject[1].

By the fundamental equations

$$d\varpi = Xdx + Ydy + Zdz - \frac{Du}{Dt}dx - \frac{Dv}{Dt}dy - \frac{Dw}{Dt}dz.$$

Now $Xdx + Ydy + Zdz = dR$, if the forces be conservative, and

$$\frac{Du}{Dt}dx + \frac{Dv}{Dt}dy + \frac{Dw}{Dt}dz$$

$$= \frac{D}{Dt}(udx + vdy + wdz) - u\frac{Ddx}{Dt} - v\frac{Ddy}{Dt} - w\frac{Ddz}{Dt},$$

in which

$$\frac{Ddx}{Dt} = d\frac{Dx}{Dt} = du, \text{ \&c.}$$

Thus, if $U^2 = u^2 + v^2 + w^2$, we have

$$d\varpi = dR - \frac{D}{Dt}(udx + vdy + wdz) + \tfrac{1}{2}dU^2 \text{ ....(1)},$$

or

$$\frac{D}{Dt}(udx + vdy + wdz) = d(R + \tfrac{1}{2}U^2 - \varpi) \text{ ....(2)}.$$

Integrating this equation along any finite arc P_1P_2, moving with the fluid, we have

$$\frac{D}{Dt}\int(udx + vdy + wdz) = (R + \tfrac{1}{2}U^2 - \varpi)_2 - (R + \tfrac{1}{2}U^2 - \varpi)_1 \text{...(3)},$$

in which suffixes denote the values of the bracketed function at the points P_2 and P_1 respectively. If the arc be a complete circuit,

$$\frac{D}{Dt}\int(udx + vdy + wdz) = 0 \text{ ....(4)};$$

[1] Vortex Motion. *Edinburgh Transactions*, 1869.

or, in words,

The line-integral of the tangential component velocity round any closed curve of a moving fluid remains constant throughout all time.

The line-integral in question is appropriately called the *circulation*, and the proposition may be stated :—

The circulation in any closed line moving with the fluid remains constant.

In a state of rest the circulation is of course zero, so that, if a fluid be set in motion by pressures transmitted from the outside or by conservative forces, the circulation along any closed line must ever remain zero, which requires that $udx + vdy + wdz$ be a complete differential.

But it does not follow conversely that in irrotational motion there can never be circulation, unless it be known that ϕ is single-valued; for otherwise $\int d\phi$ need not vanish round a closed circuit. In such a case all that can be said is that there is no circulation round any closed curve capable of being contracted to a point without passing out of space occupied by irrotationally moving fluid, or more generally, that the circulation is the same in all mutually reconcilable closed curves. Two curves are said to be reconcilable, when one can be obtained from the other by continuous deformation, without passing out of the irrotationally moving fluid.

Within an oval space, such as that included by an ellipsoid, all circuits are reconcilable, and therefore if a mass of fluid of that form move irrotationally, there can be no circulation along any closed curve drawn within it. Such spaces are called simply-connected. But in an annular space like that bounded by the surface of an anchor ring, a closed curve going round the ring is not continuously reducible to a point, and therefore there may be circulation along it, even although the motion be irrotational throughout the whole volume included. But the circulation is zero for every closed curve which does not pass round the ring, and has the same constant value for all those that do.

241. When $udx + vdy + wdz$ is an exact differential $d\phi$, the velocity in any direction is expressed by the corresponding rate of change of ϕ, which is called the velocity-potential, and

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$$

may be replaced by

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}.$$

If S denote any closed surface, the rate of flow outwards across the element dS is expressed by $\frac{d\phi}{dn}dS$, where $\frac{d\phi}{dn}$ is the rate of variation of ϕ in proceeding outwards along the normal. In the case of constant density, the total loss of fluid in time dt is thus

$$\iint \frac{d\phi}{dn} dS\,.\,dt,$$

the integration ranging over the whole surface of S. If the space S be full both at the beginning and at the end of the time dt, the loss must vanish; and thus

$$\iint \frac{d\phi}{dn} dS = 0 \quad\text{.............................(1).}$$

The application of this equation to the element $dxdydz$ gives for the equation of continuity of an incompressible fluid

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0 \quad\text{......................(2),}$$

or, as it is generally written,

$$\nabla^2\phi = 0 \quad\text{.............................(3);}$$

when it is desired to work with polar co-ordinates, the transformed equation is more readily obtained directly by applying (1) to the corresponding element of volume, than by transforming (2) in accordance with the analytical rules for effecting changes in the independent variables.

Thus, if we take polar co-ordinates in the plane xy, so that

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we find

$$\nabla^2\phi = \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \frac{1}{r^2}\frac{d^2\phi}{d\theta^2} + \frac{d^2\phi}{dz^2} \quad\text{..................(4);}$$

or, if we take polar co-ordinates in space,

$$x = r\sin\theta\cos\omega, \quad y = r\sin\theta\sin\omega, \quad z = r\cos\theta,$$

$$\nabla^2\phi = \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2\phi}{d\omega^2} \quad\text{...(5).}$$

Simpler forms are assumed in special cases, such, for example, as that of symmetry round z in (5).

When the fluid is compressible, and the motion such that the squares of small quantities may be neglected, the equation of continuity is by (3), § 238,

$$\frac{ds}{dt} + \nabla^2\phi = 0 \text{ ....(6)},$$

where any form of $\nabla^2\phi$ may be used that may be most convenient for the problem in hand.

242. The irrotational motion of incompressible fluid within any simply-connected closed space S is completely determined by the normal velocities over the surface of S. If S be a material envelope, it is evident that an arbitrary normal velocity may be impressed upon its surface, which normal velocity must be shared by the fluid immediately in contact, provided that the whole volume inclosed remain unaltered. If the fluid be previously at rest, it can acquire no molecular rotation under the operation of the fluid pressures, which shews that it must be possible to determine a function ϕ, such that $\nabla^2\phi = 0$ throughout the space inclosed by S, while over the surface $\frac{d\phi}{dn}$ has a prescribed value, limited only by the condition

$$\iint \frac{d\phi}{dn} dS = 0 \text{ ....(1)}.$$

An analytical proof of this important proposition is indicated in Thomson and Tait's *Natural Philosophy*, § 317.

There is no difficulty in proving that but one solution of the problem is possible. By Green's theorem, if $\nabla^2\phi = 0$,

$$\iiint\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) dV = \iint \phi \frac{d\phi}{dn} dS \text{ ....(2)},$$

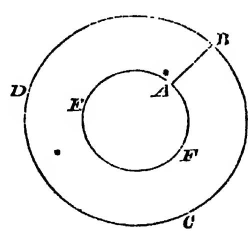
the integration on the left-hand side ranging over the volume, and on the right over the surface of S. Now if ϕ and $\phi + \Delta\phi$ be two functions, satisfying Laplace's equation, and giving prescribed surface-values of $\frac{d\phi}{dn}$, their difference $\Delta\phi$ is a function also satisfying Laplace's equation, and making $\frac{d\Delta\phi}{dn}$ vanish

over the surface of S. Under these circumstances the double integral in (2) vanishes, and we infer that at every point of S $\frac{d\Delta\phi}{dx}$, $\frac{d\Delta\phi}{dy}$, $\frac{d\Delta\phi}{dz}$ must be equal to zero. In other words $\Delta\phi$ must be constant, and the two motions identical. As a particular case, there can be no motion of the irrotational kind within the volume S, independently of a motion of the surface. The restriction to simply connected spaces is rendered necessary by the failure of Green's theorem, which, as was first pointed out by Helmholtz, is otherwise possible.

When the space S is multiply-connected, the irrotational motion is still determinate, if besides the normal velocity at every point of S there be given the values of the constant circulations in all the possible irreconcilable circuits. For a complete discussion of this question we must refer to Thomson's original memoir, and content ourselves here with the case of a doubly-connected space, which will suffice for illustration.

Let $ABCD$ be an endless tube within which fluid moves irrotationally. For this motion there must exist a velocity

Fig. 54.



potential, whose differential coefficients, expressing, as they do, the component velocities, are necessarily single-valued, but which need not itself be single-valued. The simplest way of attacking the difficulty presented by the ambiguity of ϕ, is to conceive a barrier AB taken across the ring, so as to close the passage. The space $ABCDBAEF$ is then simply continuous, and Green's theorem applies to it without modification, if allowance be made for a possible finite difference in the value of ϕ on the two sides of the barrier. This difference, if it exist, is

necessarily the same at all points of AB, and in the hydrodynamical application expresses the *circulation* round the ring.

In applying the equation

$$\iiint\left(\frac{d\phi^2}{dx^2}+\frac{d\phi^2}{dy^2}+\frac{d\phi^2}{dz^2}\right)dV=\iint\phi\frac{d\phi}{dn}dS\ldots\ldots\ldots(2),$$

we have to calculate the double integral over the two faces of the barrier as well as over the original surface of the ring. Now since $\frac{d\phi}{dn}$ has the same value on the two sides,

$$\iint\phi\frac{d\phi}{dn}dS \text{ (over two faces of } AB) =\iint\frac{d\phi}{dn}\kappa dS=\kappa\iint\frac{d\phi}{dn}dS,$$

if κ denote the constant difference of ϕ. Thus, if κ vanish, or there be no circulation round the ring, we infer, just as for a simply-connected space, that ϕ is completely determined by the surface-values of $\frac{d\phi}{dn}$. If there be circulation, ϕ is still determined, if the amount of the circulation be given. For, if ϕ and $\phi+\Delta\phi$ be two functions satisfying Laplace's equation and giving the same amount of circulation and the same normal velocities at S, their difference $\Delta\phi$ also satisfies Laplace's equation and the condition that there shall be neither circulation nor normal velocities over S. But, as we have just seen, under these circumstances $\Delta\phi$ vanishes at every point.

Although in a doubly-connected space irrotational motion is possible independently of surface normal velocities, yet such a motion cannot be generated by conservative forces nor by motions imposed (at any previous time) on the bounding surface, for we have proved that if the fluid be originally at rest, there can never be circulation along any closed curve. Hence, for multiply-connected as well as simply-connected spaces, if a fluid be set in motion by arbitrary deformation of the boundary, the whole mass comes to rest so soon as the motion of the boundary ceases.

If in a fluid moving without circulation all the fluid outside a reentrant tube-like surface of uniform section become instantaneously solid, then also at the same moment all the fluid within the tube comes to rest. This mechanical interpretation, however unpractical, will help the student to understand more

clearly what is meant by a fluid having no circulation, and it leads to an extension of Stokes' theorem with respect to molecular rotation. For, if all the fluid (moving subject to a velocity-potential) outside a spherical cavity of any radius become suddenly solid, the fluid inside the cavity can retain no motion. Or, as we may also state it, any spherical portion of an irrotationally moving fluid becoming suddenly solid would possess only a motion of translation, *without rotation*[1].

A similar proposition will apply to a circular disc, or cylinder with flat ends, in the case of fluid moving irrotationally in two dimensions only.

The motion of an incompressible fluid which has been once at rest partakes of the remarkable property (§ 79) common to that of all systems which are set in motion with prescribed velocities, namely, that the energy is the least possible. If any other motion be proposed satisfying the equation of continuity and the boundary conditions, its energy is necessarily greater than that of the motion which would be generated from rest.

243. The fact that the irrotational motion of incompressible fluid depends upon a velocity-potential satisfying Laplace's equation, is the foundation of a far-reaching analogy between the motion of such a fluid, and that of electricity or heat in a uniform conductor, which it is often of great service to bear in mind. The same may be said of the connection between all the branches of Physics which depend mathematically on a potential, for it often happens that the analogous theorems are far from equally obvious. For example, the analytical theorem that, if $\nabla^2\phi = 0$,

$$\iint \frac{d\phi}{dn}\, dS = 0$$

over a closed surface, is most readily suggested by the fluid interpretation, but once obtained may be interpreted for electric or magnetic forces.

Again, in the theory of the conduction of heat or electricity, it is obvious that there can be no steady motion in the interior of S, without transmission across some part of the bounding surface, but this, when interpreted for incompressible fluids, gives an important and rather recondite law.

[1] Thomson on *Vortex Motion*, *loc. cit.*

244. When a velocity-potential exists, the equation to determine the pressure may be put into a simpler form. We have from (1), § 240,

$$d\varpi = dR - \frac{D}{Dt}d\phi + \tfrac{1}{2}dU^2 \text{....................(1)},$$

whence by integration

$$\varpi = \int \frac{dp}{\rho} = R - \frac{D\phi}{Dt} + \tfrac{1}{2}U^2.$$

Now
$$\frac{D\phi}{Dt} = \frac{d\phi}{dt} + u^2 + v^2 + w^2;$$

so that

$$\int \frac{dp}{\rho} = R - \frac{d\phi}{dt} - \tfrac{1}{2}U^2 \text{....................(2)},$$

which is the form ordinarily given.

If ρ be constant, $\int \frac{dp}{\rho}$ is replaced, of course, by $\frac{p}{\rho}$.

The relation between p and ϕ in the case of impulsive motion from rest may be deduced from (2) by integration. We see that

$$\frac{1}{\rho}\int p\,dt = -\phi \quad \text{ultimately.}$$

The same conclusion may be arrived at by a direct application of mechanical principles to the circumstances of impulsive motion.

If $p = \kappa\rho$, equation (2) takes the form

$$\kappa \log \rho = R - \frac{d\phi}{dt} - \tfrac{1}{2}U^2 \text{....................(3)}.$$

If the motion be such that the component velocities are always the same at the same point of space, it is called *steady*, and ϕ becomes independent of the time. The equation of pressure is then

$$\int \frac{dp}{\rho} = R - \tfrac{1}{2}U^2 \text{.............................(4)},$$

or in the case when there are no impressed forces,

$$\int \frac{dp}{\rho} = C - \tfrac{1}{2}U^2 \text{.............................(5)}.$$

In most acoustical applications of (2), the velocities and condensation are small; and then we may neglect the term $\frac{1}{2}U^2$, and substitute $\frac{\delta p}{\rho_0}$ for $\int \frac{dp}{\rho}$, if δp denote the small variable part of p; thus

$$\frac{\delta p}{\rho_0} = R - \frac{d\phi}{dt} \quad\text{.............................}(6),$$

which with

$$\frac{ds}{dt} + \nabla^2\phi = 0 \quad\text{.............................}(7)$$

are the equations by means of which the small vibrations of an elastic fluid are to be investigated.

If $a^2 = \frac{dp}{d\rho}$, so that $\delta p = a^2\rho_0 s$, (6) becomes

$$a^2 s = R - \frac{d\phi}{dt}, \quad\text{.............................}(8),$$

and we get on elimination of s,

$$\frac{d^2\phi}{dt^2} = \frac{dR}{dt} + a^2\nabla^2\phi \quad\text{.............................}(9).$$

245. The simplest kind of wave-motion is that in which the excursions of every particle are parallel to a fixed line, and are the same in all planes perpendicular to that line. Let us therefore (assuming that $R = 0$) suppose that ϕ is a function of x (and t) only. Our equation (9) § 244 becomes

$$\frac{d^2\phi}{dt^2} = a^2\frac{d^2\phi}{dx^2} \quad\text{.............................}(1),$$

the same as that already considered in the chapter on Strings. We there found that the general solution is

$$\phi = f(x - at) + F(x + at) \quad\text{.............................}(2),$$

representing the propagation of independent waves in the positive and negative directions with the common velocity a.

Within such limits as allow the application of the approximate equation (1), the velocity of sound is entirely independent of the form of the wave, being, for example, the same for simple waves

$$\phi = A\cos\frac{2\pi}{\lambda}(x - at),$$

whatever the wave-length may be. The condition satisfied by the positive wave, and therefore by the initial disturbance if a positive wave alone be generated, is

$$a\frac{d\phi}{dx} + \frac{d\phi}{dt} = 0,$$

or by (8) § 244

$$u - as = 0 \quad\text{.......................(3).}$$

Similarly, for a negative wave

$$u + as = 0 \quad\text{.......................(4).}$$

Whatever the initial disturbance may be (and u and s are both arbitrary), it can always be divided into two parts, satisfying respectively (3) and (4), which are propagated undisturbed. In each component wave the direction of propagation is the same as that of the motion of the *condensed* parts of the fluid.

The rate at which energy is transmitted across unit of area of a plane parallel to the front of a progressive wave may be regarded as the mechanical measure of the intensity of the radiation. In the case of a simple wave, for which

$$\phi = A \cos \frac{2\pi}{\lambda}(x - at) \quad\text{......................(5),}$$

the velocity $\dot{\xi}$ of the particle at x $\left(\text{equal to } \frac{d\phi}{dx}\right)$ is given by

$$\dot{\xi} = -\frac{2\pi}{\lambda} A \sin \frac{2\pi}{\lambda}(x - at) \quad\text{................(6),}$$

and the displacement ξ is given by

$$\xi = -\frac{A}{a} \cos \frac{2\pi}{\lambda}(x - at) \quad\text{..................(7).}$$

The pressure $p = p_0 + \delta p$, where by (6) § 244

$$\delta p = -\frac{2\pi}{\lambda} \rho_0 aA \sin \frac{2\pi}{\lambda}(x - at) \quad\text{..........(8).}$$

Hence, if W denote the work transmitted across unit area of the plane x in time t,

$$\frac{dW}{dt} = (p_0 + \delta p)\,\dot{\xi} = \tfrac{1}{2}\rho_0 a \left(\frac{2\pi}{\lambda}\right)^2 A^2 + \text{periodic terms.}$$

If the integration with respect to time extend over any number of complete periods, or practically whenever its range is sufficiently long, the periodic terms may be omitted, and we may take

$$W : t = \tfrac{1}{2}\rho_0 a \left(\frac{2\pi}{\lambda}\right)^2 A^2 \quad\text{....................(9),}$$

or by (6), if β denote the maximum value of $\dot{\xi}$,

$$W = \tfrac{1}{2}\rho_0\beta^2 a\,t \qquad\qquad (10).$$

Thus the work consumed in generating waves of harmonic type is the same as would be required to give the maximum velocity β to the whole mass of air through which the waves extend[1].

In terms of the maximum excursion α by (7) and (9)

$$W = 2\pi^2\rho_0\frac{a^3}{\lambda^2}\alpha^2 t = 2\pi^2\rho_0 a t\frac{\alpha^2}{\tau^2} \qquad\qquad (11)[2],$$

where $\tau\,(=\lambda \div a)$ is the periodic time. In a *given medium* the mechanical measure of the intensity is proportional to the square of the amplitude directly, and to the square of the periodic time inversely. The reader, however, must be on his guard against supposing that the mechanical measure of intensity of undulations of different wave lengths is a proper measure of the loudness of the corresponding sounds, as perceived by the ear.

In any plane progressive wave, whether the type be harmonic or not, the whole energy is equally divided between the potential and kinetic forms. Perhaps the simplest road to this result is to consider the formation of positive and negative waves from an initial disturbance, whose energy is wholly potential[3]. The total energies of the two derived progressive waves are evidently equal, and make up together the energy of the original disturbance. Moreover, in each progressive wave the condensation (or rarefaction) is one-half of that which existed at the corresponding point initially, so that the *potential* energy of each progressive wave is *one-quarter* of that of the original disturbance. Since, as we have just seen, the *whole* energy is *one-half* of the same quantity, it follows that in a progressive wave of any type one-half of the energy is potential and one-half is kinetic.

The same conclusion may also be drawn from the general expressions for the potential and kinetic energies and the relations between velocity and condensation expressed in (3) and (4). The potential energy of the element of volume dV is the work

[1] The earliest statement of the principle embodied in equation (10) that I have met with is in a paper by Sir W. Thomson, "On the possible density of the luminiferous medium, and on the mechanical value of a cubic mile of sun-light." *Phil. Mag.* IX. p. 36. 1855.

[2] Bosanquet. *Phil. Mag.* XLV. p. 173. 1873.

[3] *Phil. Mag.* (5) I. p. 260. 1876.

that would be gained during the expansion of the corresponding quantity of gas from its actual to its normal volume, the expansion being opposed throughout by the normal pressure p_0. At any stage of the expansion, when the condensation is s', the effective pressure δp is by § 244 $a^2 \rho_0 s'$, which pressure has to be multiplied by the corresponding increment of volume $dV.ds'$. The whole work gained during the expansion from dV to $dV(1+s)$ is therefore $a^2\rho_0 dV.\int_0 s'ds'$ or $\frac{1}{2}a^2\rho_0 dV.s^2$. The general expressions for the potential and kinetic energies are accordingly

$$\text{potential energy} = \tfrac{1}{2}a^2\rho_0\iiint s^2 dV \quad\ldots\ldots\ldots\ldots(12),$$

$$\text{kinetic energy} = \tfrac{1}{2}\rho_0\iiint u^2 dV \quad\ldots\ldots\ldots\ldots(13),$$

and these are equal in the case of plane progressive waves for which

$$u = \pm as.$$

If the plane progressive waves be of harmonic type, u and s at any moment of time are circular functions of one of the space co-ordinates (x), and therefore the mean value of their squares is one-half of the maximum value. Hence the total energy of the waves is equal to the kinetic energy of the whole mass of air concerned, moving with the maximum velocity to be found in the waves, or to the potential energy of the same mass of air when condensed to the maximum density of the waves.

246. The first theoretical investigation of the velocity of sound was made by Newton, who assumed that the relation between pressure and density was that formulated in Boyle's law. If we assume $p = \kappa\rho$, we see that the velocity of sound is expressed by $\sqrt{\kappa}$, or $\sqrt{p} \div \sqrt{\rho}$, in which the dimensions of p (= force ÷ area) are $[M]\,[L]^{-1}\,[T]^{-2}$, and those of ρ (= mass ÷ volume) are $[M]\,[L]^{-3}$. Newton expressed the result in terms of the '*height of the homogeneous atmosphere*,' defined by the equation

$$g\rho h = p \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where p and ρ refer to the pressure and the density at the earth's surface. The velocity of sound is thus $\sqrt{gh}$, or the velocity which would be acquired by a body falling freely under the action of gravity through half the height of the homogeneous atmosphere.

To obtain a numerical result we require to know a pair of simultaneous values of p and ρ. It is found by experiment that at 0° Cent. under a pressure of 1033 grammes per square centimetre, the density of dry air is ·001293 grammes per cubic centimetre. If we take the centimetre, gramme, and second as the fundamental units the (C.G.S. system), these data give

$$p = 1033 \times g = 1033 \times 981, \qquad \rho = \cdot 001293,$$

whence

$$a = \sqrt{\frac{1033 \times 981}{\cdot 001293}} = 27995;$$

so that the velocity of sound at 0° would be 279·95 metres per second, falling short of the result of direct observation by about a sixth part.

Newton's investigation established that the velocity of sound should be independent of the amplitude of the vibration, and also of the pitch, but the discrepancy between his calculated value (published in 1687) and the experimental value was not explained until Laplace pointed out that the use of Boyle's law involved the assumption that in the condensations and rarefactions accompanying sound the temperature remains constant, in contradiction to the known fact that, when air is suddenly compressed, its temperature rises. The laws of Boyle and Charles supply only one relation between the three quantities, pressure, volume, and temperature, of a gas, viz.

$$pv = R\theta \qquad (2),$$

where the temperature θ is measured from the zero of the gas thermometer; and therefore without some auxiliary assumption it is impossible to specify the connection between p and v (or ρ). Laplace considered that the condensations and rarefactions concerned in the propagation of sound take place with such rapidity that the heat and cold produced have not time to pass away, and that therefore the relation between volume and pressure is sensibly the same as if the air were confined in an absolutely non-conducting vessel. Under these circumstances the change of pressure corresponding to a given condensation or rarefaction is greater than on the hypothesis of constant temperature, and the velocity of sound is accordingly increased.

In equation (2) let v denote the volume and p the pressure of the unit of mass, and let θ be expressed in centigrade degrees

reckoned from the absolute zero[1]. The condition of the gas (if uniform) is defined by any two of the three quantities p, v, θ, and the third may be expressed in terms of them. The relation between the simultaneous variations of the three quantities is

$$\frac{d\theta}{\theta}=\frac{dp}{p}+\frac{dv}{v} \quad\text{.........................(3).}$$

In order to effect the change specified by dp and dv, it is in general necessary to communicate heat to the gas. Calling the necessary quantity of heat dQ, we may write

$$dQ=\left(\frac{dQ}{dv}\right)dv+\left(\frac{dQ}{dp}\right)dp \quad\text{................(4).}$$

Suppose now (a) that $dp=0$. Equations (3) and (4) give

$$\frac{dQ}{d\theta}\,(p\text{ const.})=\left(\frac{dQ}{dv}\right)\frac{v}{\theta},$$

where $\frac{dQ}{d\theta}\,(p\text{ const.})$ expresses the specific heat of the gas under a constant pressure. This being denoted by κ_p, we have

$$\kappa_p=\left(\frac{dQ}{dv}\right)\frac{v}{\theta} \quad\text{.........................(5).}$$

Again, suppose (b) that $dv=0$. We find in a similar manner that, if κ_v denote the specific heat under a constant volume,

$$\kappa_v=\left(\frac{dQ}{dp}\right)\frac{p}{\theta} \quad\text{.........................(6).}$$

In order to obtain the relation between dp and dv when there is no communication of heat, we have only to put $dQ=0$. Thus

$$\left(\frac{dQ}{dv}\right)dv+\left(\frac{dQ}{dp}\right)dp=0,$$

or, on substituting for the differential coefficients of Q their values in terms of κ_v, κ_p,

$$\kappa_p\frac{dv}{v}+\kappa_v\frac{dp}{p}=0 \quad\text{....................(7).}$$

Since $v=\frac{1}{\rho}$, $\qquad \frac{dv}{v}=-\frac{d\rho}{\rho}$;

so that

$$a^2=\frac{dp}{d\rho}=\frac{p}{\rho}\frac{\kappa_p}{\kappa_v}=\frac{p}{\rho}\gamma \quad\text{.....................(8),}$$

[1] On the ordinary centigrade scale the absolute zero is about $-273°$.

if, as usual, the ratio of the specific heats be denoted by γ. Laplace's value of the velocity of sound is therefore greater than Newton's in the ratio of $\sqrt{\gamma} : 1$.

By integration of (8), we obtain for the relation between p and ρ, on the supposition of no communication of heat,

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \quad \text{.........................(9)[1]},$$

where p_0, ρ_0 are two simultaneous values. Under the same circumstances the relation between pressure and temperature is by (3)

$$\frac{p}{p_0} = \left(\frac{\theta}{\theta_0}\right)^{\frac{\gamma}{\gamma-1}} \quad \text{.........................(10)}.$$

The magnitude of γ cannot be determined with accuracy by direct experiment, but an approximate value may be obtained by a method of which the following is the principle. Air is compressed into a reservoir capable of being put into communication with the external atmosphere by opening a wide valve. At first the temperature of the compressed air is raised, but after a time the superfluous heat passes away and the whole mass assumes the temperature of the atmosphere Θ. Let the pressure (measured by a manometer) be p. The valve is now opened for as short a time as is sufficient to permit the equilibrium of pressure to be completely established, that is, until the internal pressure has become equal to that of the atmosphere P. If the experiment be properly arranged, this operation is so quick that the air in the vessel has not sufficient time to receive heat from the sides, and therefore expands nearly according to the law expressed in (9). Its temperature θ at the moment the operation is complete is therefore determined by

$$\frac{p}{P} = \left(\frac{\Theta}{\theta}\right)^{\frac{\gamma}{\gamma-1}} \quad \text{.........................(11)}.$$

The enclosed air is next allowed to absorb heat until it has regained the atmospheric temperature Θ, and its pressure (p') is then observed. During the last change the volume is constant, and therefore the relation between pressure and temperature gives

$$\frac{P}{p'} = \frac{\theta}{\Theta} \quad \text{.........................(12)};$$

[1] It is here assumed that γ is constant. This equation appears to have been given first by Poisson.

so that by elimination of $\theta : \Theta$,

$$\frac{p}{P} = \left(\frac{p'}{P}\right)^{\frac{\gamma}{\gamma-1}},$$

whence

$$\gamma = \frac{\log p - \log P}{\log p - \log p'} \quad \text{..................} (13).$$

By experiments of this nature Clement and Desormes determined $\gamma = 1{\cdot}3492$; but the method is obviously not susceptible of any great accuracy. The value of γ required to reconcile the calculated and observed velocities of sound is 1·408, of the substantial correctness of which there can be little doubt.

We are not, however, dependent on the phenomena of sound for our knowledge of the magnitude of γ. The value of κ_p—the specific heat at constant pressure—has been determined experimentally by Regnault; and although on account of inherent difficulties the experimental method may fail to yield a satisfactory result for κ_v, the information sought for may be obtained indirectly by means of a relation between the two specific heats, brought to light by the modern science of Thermodynamics.

If from the equations

$$\left.\begin{aligned} \frac{dQ}{\theta} &= \kappa_p \frac{dv}{v} + \kappa_v \frac{dp}{p} \\ \frac{d\theta}{\theta} &= \quad \frac{dv}{v} + \quad \frac{dp}{p} \end{aligned}\right\} \quad \text{..................} (14)$$

we eliminate dp, there results

$$dQ = (\kappa_p - \kappa_v)\frac{pdv}{R} + \kappa_v d\theta \quad \text{................} (15).$$

Let us suppose that $dQ = 0$, or that there is no communication of heat. It is known that the heat developed during the compression of an approximately perfect gas, such as air, is almost exactly the thermal equivalent of the work done in compressing it. This important principle was assumed by Mayer in his celebrated memoir on the dynamical theory of heat, though on grounds which can hardly be considered adequate. However that may be, the principle itself is very nearly true, as has since been proved by the experiments of Joule and Thomson.

If we measure heat in dynamical units, Mayer's principle may be expressed, $\kappa_v d\theta = pdv$ on the understanding that there

is no communication of heat. Comparing this with (15), we see that

$$\kappa_p - \kappa_v = R \text{ ....(16)},$$

and therefore

$$\gamma = \frac{\kappa_p}{\kappa_v} = \frac{\kappa_p}{\kappa_p - R} \text{ ....(17)}.$$

The value of pv in gravitation measure (gramme, centimetre) is, as we saw, $1033 \div \cdot 001293$, at 0° Cent. so that

$$R = \frac{1033}{\cdot 001293 \times 272 \cdot 85}.$$

By Regnault's experiments the specific heat of air is ·2379 of that of water; and in order to raise a gramme of water one degree Cent., 42350 gramme-centimetres of work must be done on it. Hence with the same units as for R,

$$\kappa_p = \cdot 2379 \times 42350.$$

Calculating from these data, we find $\gamma = 1 \cdot 410$, agreeing almost exactly with the value deduced from the velocity of sound. This investigation is due to Rankine, who employed it in 1850 to calculate the specific heat of air, taking Joule's equivalent and the observed velocity of sound as data. In this way he anticipated the result of Regnault's experiments, which were not published until 1853.

247. Laplace's theory has often been the subject of misapprehension among students, and a stumblingblock to those remarkable persons, called by De Morgan, 'paradoxers.' But there can be no reasonable doubt that, antecedently to all calculation, the hypothesis of no communication of heat is greatly to be preferred to the equally special hypothesis of constant temperature. There would be a real difficulty if the velocity of sound were not decidedly in excess of Newton's value, and the wonder is rather that the cause of the excess remained so long undiscovered.

The only question which can possibly be considered open, is whether a small part of the heat and cold developed may not escape by conduction or radiation before producing its full effect. Everything must depend on the rapidity of the alternations. Below a certain limit of slowness, the heat in excess, or defect, would have time to adjust itself, and the temperature would remain sensibly constant. In this case the relation between

pressure and density would be that which leads to Newton's value of the velocity of sound. On the other hand, above a certain limit of quickness, the gas would behave as if confined in a non-conducting vessel, as supposed in Laplace's theory. Now although the circumstances of the actual problem are better represented by the latter than by the former supposition, there may still (it may be said) be a sensible deviation from the law of pressure and density involved in Laplace's theory, entailing a somewhat slower velocity of propagation of sound. This question has been carefully discussed by Stokes in a paper published in 1851[1], of which the following is an outline.

The mechanical equations for the *small* motion of air are

$$\frac{dp}{dx} = -\rho\frac{du}{dt}\ \&c. \qquad\qquad (1),$$

with the equation of continuity

$$\frac{ds}{dt} + \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \qquad\qquad (2).$$

The temperature is supposed to be uniform except in so far as it is disturbed by the vibrations themselves, so that if θ denote the *excess* of temperature,

$$p = \kappa\rho\,(1 + s + \alpha\theta) \qquad\qquad (3).$$

The effect of a small sudden condensation s is to produce an elevation of temperature, which may be denoted by βs. Let dQ be the quantity of heat entering the element of volume in time dt, measured by the rise of temperature that it would produce, if there were no condensation. Then (the distinction between $\frac{D}{Dt}$ and $\frac{d}{dt}$ being neglected)

$$\frac{d\theta}{dt} = \beta\frac{ds}{dt} + \frac{dQ}{dt} \qquad\qquad (4),$$

$\frac{dQ}{dt}$ being a function of θ and its differential coefficients with respect to space, dependent on the special character of the dissipation. Two extreme cases may be mentioned; the first when the tendency to equalisation of temperature is due to conduction, the second when the operating cause is radiation, and the transparency of the medium such that radiant heat is

[1] *Phil. Mag.* (4) I. 305.

not sensibly absorbed within a distance of several wave-lengths. In the former case $\frac{dQ}{dt} \propto \nabla^2\theta$, and in the latter, which is that selected by Stokes for analytical investigation, $\frac{dQ}{dt} \propto (-\theta)$, Newton's law of radiation being assumed as a sufficient approximation to the truth. We have then

$$\frac{d\theta}{dt} = \beta \frac{ds}{dt} - q\theta \text{......................(5).}$$

In the case of plane waves, to which we shall confine our attention, v and w vanish, while u, p, s, θ are functions of x (and t) only. Eliminating p and u between (1), (2) and (3), we find

$$\frac{d^2s}{dt^2} = \kappa\left(\frac{d^2s}{dx^2} + \alpha\frac{d^2\theta}{dx^2}\right),$$

from which and (5) we get

$$\left(\frac{d}{dt} + q\right)\frac{d^2s}{dt^2} = \kappa\left(\gamma\frac{d}{dt} + q\right)\frac{d^2s}{dx^2} \text{..............(6),}$$

if γ be written (in the same sense as before) for $1 + \alpha\beta$.

If the vibrations be harmonic, we may suppose that s varies as e^{int}, and the equation becomes

$$\frac{d^2s}{dx^2} + \frac{n^2}{\kappa} \cdot \frac{q+in}{q+i\gamma n} \cdot s = 0 \text{..................(7).}$$

Let the coefficient of s in (7) be put into the form $\mu^2 e^{-2i\psi}$, where

$$\mu^4 = \frac{n^4}{\kappa^2} \cdot \frac{q^2+n^2}{q^2+\gamma^2n^2} \text{......................(8),}$$

and

$$2\psi = \tan^{-1}\frac{\gamma n}{q} - \tan^{-1}\frac{n}{q} = \tan^{-1}\frac{(\gamma-1)\,nq}{\gamma n^2+q^2} \text{........(9).}$$

Equation (7) is then satisfied by terms of the form

$$e^{\pm i\mu(\cos\psi - i\sin\psi)x},$$

but (μ being positive, and ψ less than $\frac{1}{2}\pi$) if we wish for the expression of the wave travelling in the positive direction, we must take the lower sign. Discarding the imaginary part, we find as the appropriate solution

$$s = Ae^{-\mu\sin\psi\, x}\cos(nt - \mu\cos\psi\, x) \text{..........(10).}$$

The first thing to be noticed is that the sound cannot be propagated to a distance unless $\sin\psi$ be insensible.

The velocity of propagation (V) is

$$V = n\mu^{-1}\sec\psi \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

which, when $\sin\psi$ is insensible, reduces to

$$V = n\mu^{-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Now from (9) we see that ψ cannot be insensible, unless $q : n$ is either very great, or very small. On the first supposition from (11), or directly from (7), we have approximately, $V = \sqrt{\kappa}$ (Newton); and on the second, $V = \sqrt{\kappa\gamma}$, (Laplace), as ought evidently to be the case, when the meaning of q in (5) is considered. What we now learn is that, if q and n were comparable, the effect would be not merely a deviation of V from either of the limiting values, but a rapid stifling of the sound, which we know does not take place in nature.

Of this theoretical result we may convince ourselves, as Stokes explains, without the use of analysis. Imagine a mass of air to be confined within a closed cylinder, in which a piston is worked with a reciprocating motion. If the period of the motion be very long, the temperature of the air remains nearly constant, the heat developed by compression having time to escape by conduction or radiation. Under these circumstances the pressure is a function of volume, and whatever work has to be expended in producing a given compression is refunded when the piston passes through the same position in the reverse direction; no work is consumed in the long run. Next suppose that the motion is so rapid that there is no time for the heat and cold developed by the condensations and rarefactions to escape. The pressure is still a function of volume, and no work is dissipated. The only difference is that now the variations of pressure are more considerable than before in comparison with the variations of volume. We see how it is that both on Newton's and on Laplace's hypothesis, the waves travel without dissipation, though with different velocities.

But in intermediate cases, when the motion of the piston is neither so slow that the temperature remains constant nor so quick that the heat has no time to adjust itself, the result is different. The work expended in producing a small condensa-

tion is no longer completely refunded during the corresponding rarefaction on account of the diminished temperature, part of the heat developed by the compression having in the meantime escaped. In fact the passage of heat by conduction or radiation from a warmer to a finitely colder body always involves dissipation, a principle which occupies a fundamental position in the science of Thermodynamics. In order therefore to maintain the motion of the piston, energy must be supplied from without, and if there be only a limited store to be drawn from, the motion must ultimately subside.

Another point to be noticed is that, if q and n were comparable, V would depend upon n, viz. on the pitch of the sound, a state of things which from experiment we have no reason to suspect. On the contrary the evidence of observation goes to prove that there is no such connection.

From (10) we see that the falling off in the intensity, estimated per wave-length, is a maximum with $\tan\psi$, or ψ; and by (9) ψ is a maximum, when $q : n = \sqrt{\gamma}$. In this case

$$\mu = n\kappa^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}, \qquad 2\psi = \tan^{-1}\gamma^{\frac{1}{2}} - \tan^{-1}\gamma^{-\frac{1}{2}} \text{......}(13),$$

whence, if we take $\gamma = 1{\cdot}36$, $2\psi = 8^\circ\, 47'$.

Calculating from these data, we find that for each wave-length of advance, the amplitude of the vibration would be diminished in the ratio ·6172.

To take a numerical example, let

$$\tau = \frac{1}{300} \text{ of a second,} \quad \lambda = \text{wave-length} = 44 \text{ inches.}$$

In 20 yards the intensity would be diminished in the ratio of about 7 millions to one.

Corresponding to this,

$$q = 2198 \text{..........................}(14).$$

If the value of q were actually that just written, sounds of the pitch in question would be very rapidly stifled. We therefore infer that q is in fact either much greater or else much less. But even so large a value as 2000 is utterly inadmissible, as we may convince ourselves by considering the significance of equation (5).

Suppose that by a rigid envelope transparent to radiant heat, the volume of a small mass of gas were maintained constant, then the equation to determine its thermal condition at any time is

$$\frac{d\theta}{dt} + q\theta = 0,$$

whence

$$\theta = Ae^{-qt} \qquad \text{.........................(15)},$$

where A denotes the initial excess of temperature, proving that after a time q^{-1} the excess of temperature would fall to less than half its original value. To suppose that this could happen in a two thousandth of a second of time would be in contradiction to the most superficial observation.

We are therefore justified in assuming that q is very small in comparison with n, and our equations then become approximately

$$\mu = \frac{n}{\kappa^{\frac{1}{2}}\gamma^{\frac{1}{2}}}, \qquad 2\psi = \frac{\gamma - 1}{\gamma}\frac{q}{n}, \qquad V = n\mu^{-1} = \kappa^{\frac{1}{2}}\gamma^{\frac{1}{2}},$$

$$s = Ae^{-(1-\gamma^{-1})\frac{qx}{2V}} \cos\frac{2\pi}{\lambda}(Vt - x) \qquad \text{..........(16)}.$$

The effects of a small radiation of heat are to be sought for rather in a damping of the vibration than in an altered velocity of propagation.

Stokes calculates that if $\gamma = 1{\cdot}414$, $V = 1100$, the ratio $(N : 1)$ in which the intensity is diminished in passing over a distance x, is given by $\log_{10} N = {\cdot}0001156\, qx$ in foot-second measure. Although we are not able to make precise measurements of the intensity of sound, yet the fact that audible vibrations can be propagated for many miles excludes any such value of q as could appreciably affect the velocity of transmission.

Neither is it possible to attribute to the air such a conducting power as could materially disturb the application of Laplace's theory. In order to trace the effects of conduction, we have only to replace q in (5) by $-q'\frac{d^2}{dx^2}$. Assuming as a particular solution

$$s = Ae^{i(nt+mx)},$$

we find

$$m^2 in\kappa\gamma = in^3 + q'n^2m^2 - \kappa q'm^4,$$

whence, if q' be relatively small,

$$m = \frac{-n}{\sqrt{\kappa\gamma}}\left(1 - \frac{\gamma - 1}{\gamma}\frac{q'n}{2\kappa\gamma}\,i\right) \ldots\ldots\ldots\ldots\ldots(17).$$

Thus the solution in real quantities is

$$s = A \,.\, Exp\left(-\frac{\gamma-1}{\gamma}\frac{q'n^2x}{2(\kappa\gamma)^{\frac{3}{2}}}\right).\cos\left(nt - \frac{nx}{\sqrt{\kappa\gamma}}\right)\ldots\ldots(18),$$

leaving the velocity of propagation to this order of approximation still equal to $\sqrt{\kappa\gamma}$.

From (18) it appears that the first effect of conduction, as of radiation, is on the amplitude rather than on the velocity of propagation. In truth the conducting power of gases is so feeble, and in the case of audible sounds at any rate the time during which conduction can take place is so short, that disturbance from this cause is not to be looked for.

In the preceding discussions the waves are supposed to be propagated in an open space. When the air is confined within a tube, whose diameter is small in comparison with the wave-length, the conditions of the problem are altered, at least in the case of conduction. What we have to say on this head will, however, come more conveniently in another place.

248. From the expression $\sqrt{(p\gamma)} \div \sqrt{\rho}$, we see that in the same gas the velocity of sound is independent of the density, because if the temperature be constant, p varies as ρ $(p = R\rho\theta)$. On the other hand the velocity of sound is proportional to the square root of the absolute temperature, so that if a_0 be its value at 0° Cent.

$$a = a_0\sqrt{1 + \frac{\theta'}{273}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where the temperature is measured in the ordinary manner from the freezing point of water.

The most conspicuous effect of the dependence of the velocity of sound on temperature is the variability of the pitch of organ pipes. We shall see in the following chapters that the period of the note of a flue organ-pipe is the time occupied by a pulse in running over a distance which is a definite multiple of the length of the pipe, and therefore varies inversely as the velocity of propagation. The inconvenience arising from this alteration

of pitch is aggravated by the fact that the reed pipes are not similarly affected; so that a change of temperature puts an organ out of tune with itself.

Prof. Mayer[1] has proposed to make the connection between temperature and wave-length the foundation of a pyrometric method, but I am not aware whether the experiment has ever been carried out.

The correctness of (1) as regards air at the temperatures of 0° and 100° has been verified experimentally by Kundt. See § 260.

In different gases at given temperature and pressure a is inversely proportional to the square roots of the densities, at least if γ be constant[2]. For the non-condensable gases γ does not sensibly vary from its value for air.

The velocity of sound is not entirely independent of the degree of dryness of the air, since at a given pressure moist air is somewhat lighter than dry air. It is calculated that at 50° F., air saturated with moisture would propagate sound between 2 and 3 feet per second faster than if it were perfectly dry.

The formula $a^2 = \frac{dp}{d\rho}$ may be applied to calculate the velocity of sound in liquids, or, if that be known, to infer conversely the coefficient of compressibility. In the case of water it is found by experiment that the compression per atmosphere is ·0000457. Thus, if $dp = 1038 \times 981$ in absolute C.G.S. units,

$$d\rho = \cdot 0000457, \quad \text{since } \rho = 1.$$

Hence

$$a = 1489 \text{ metres per second,}$$

which does not differ much from the observed value (1435).

249. In the preceding sections the theory of plane waves has been derived from the general equations of motion. We now proceed to an independent investigation in which the motion is expressed in terms of the actual position of the layers of air instead of by means of the velocity potential, whose aid is no longer necessary inasmuch as in one dimension there can be no question of molecular rotation.

[1] On an Acoustic Pyrometer. *Phil. Mag.* XLV. p. 18. 1873.

[2] According to the kinetic theory of gases, the velocity of sound is determined solely by, and is proportional to, the mean velocity of the molecules. Preston, *Phil. Mag.* (5) III. p. 441. 1877.

If y, $y+\frac{dy}{dx}dx$, define the actual positions at time t of neighbouring layers of air whose equilibrium positions are defined by x and $x+dx$, the density ρ of the included slice is given by

$$\rho : \rho_0 = 1 : \frac{dy}{dx} \quad\text{.........................(1)},$$

whence by (9) § 246,

$$p : p_0 = 1 : \left(\frac{dy}{dx}\right)^{\gamma} \quad\text{....................(2)},$$

the expansions and condensations being supposed to take place according to the adiabatic law. The mass of unit of area of the slice is $\rho_0 dx$, and the corresponding moving force is $-\frac{dp}{dx}dx$, giving for the equation of motion

$$\rho_0 \frac{d^2y}{dt^2} + \frac{dp}{dx} = 0 \quad\text{..................(3)}.$$

Between (2) and (3) p is to be eliminated. Thus,

$$\left(\frac{dy}{dx}\right)^{\gamma+1}\frac{d^2y}{dt^2} = \frac{p_0\gamma}{\rho_0}\frac{d^2y}{dx^2} \quad\text{....................(4)}.$$

Equation (4) is an *exact* equation defining the actual abscissa y in terms of the equilibrium abscissa x and the time. If the motion be assumed to be small, we may replace $\left(\frac{dy}{dx}\right)^{\gamma+1}$, which occurs as the coefficient of the small quantity $\frac{d^2y}{dt^2}$, by its approximate value unity; and (4) then becomes

$$\frac{d^2y}{dt^2} = \frac{p_0\gamma}{\rho_0}\frac{d^2y}{dx^2} \quad\text{.......................(5)},$$

the ordinary approximate equation.

If the expansion be isothermal, as in Newton's theory, the equations corresponding to (4) and (5) are obtained by merely putting $\gamma = 1$.

Whatever may be the relation between p and ρ, depending on the constitution of the medium, the equation of motion is by (1) and (3)

$$\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dt^2} = \frac{dp}{d\rho}\frac{d^2y}{dx^2} \quad\text{....................(6)},$$

from which ρ, occurring in $\frac{dp}{d\rho}$, is to be eliminated by means of the relation between ρ and $\frac{dy}{dx}$ expressed in (1).

250. In the preceding investigations of aerial waves we have supposed that the air is at rest except in so far as it is disturbed by the vibrations of sound, but we are of course at liberty to attribute to the whole mass of air concerned any common motion. If we suppose that the air is moving in the direction contrary to that of the waves and with the same actual velocity, the wave form, if permanent, is stationary in space, and the motion is *steady*. In the present section we will consider the problem under this aspect, as it is important to obtain all possible clearness in our views on the mechanics of wave propagation.

If u_0, p_0, ρ_0 denote respectively the velocity, pressure, and density of the fluid in its undisturbed state, and if u, p, ρ be the corresponding quantities at a point in the wave, we have for the equation of continuity

$$\rho u = \rho_0 u_0 \quad\text{.........................(1)},$$

and by (5) § 244 for the equation of energy

$$\int_{p_0}^{p} \frac{dp}{\rho} = \tfrac{1}{2}u_0^2 - \tfrac{1}{2}u^2 \quad\text{.....................(2)}.$$

Eliminating u, we get

$$\int_{p_0}^{p} \frac{dp}{\rho} = \tfrac{1}{2}u_0^2\left(1 - \frac{\rho_0^2}{\rho^2}\right) \quad\text{.....................(3)},$$

determining the law of pressure under which alone it is possible for a stationary wave to maintain itself in fluid moving with velocity u_0. From (3)

$$\frac{dp}{d\rho} = u_0^2\frac{\rho_0^2}{\rho^2} \quad\text{.........................(4)},$$

or

$$p = \text{constant} - \frac{u_0^2\rho_0^2}{\rho} \quad\text{.....................(5)}.$$

Since the relation between the pressure and the density of actual gases is not that expressed in (5), we conclude that a self-maintaining stationary aerial wave is an impossibility, whatever

may be the velocity u_0 of the general current, or in other words that a wave cannot be propagated relatively to the undisturbed parts of the gas without undergoing an alteration of type. Nevertheless, when the changes of density concerned are small, (5) may be satisfied approximately; and we see from (4) that the velocity of stream necessary to keep the wave stationary is given by

$$u_0 = \sqrt{\left(\frac{dp}{d\rho}\right)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

which is the same as the velocity of the wave estimated relatively to the fluid.

This method of regarding the subject shews, perhaps more clearly than any other, the nature of the relation between velocity and condensation § 245 (3), (4). In a stationary wave form a loss of velocity accompanies an augmented density according to the principle of energy, and therefore the fluid composing the condensed parts of a wave moves forward more slowly than the undisturbed portions. Relatively to the fluid therefore the motion of the condensed parts is in the same direction as that in which the waves are propagated.

When the relation between pressure and density is other than that expressed in (5), a stationary wave can be maintained only by the aid of an impressed force. By (1) and (2) § 237 we have, on the supposition that the motion is steady,

$$X = u\frac{du}{dx} + \frac{1}{\rho}\frac{dp}{dx} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

while the relation between u and ρ is given by (1). If we suppose that $p = a^2\rho$, (7) becomes

$$X = (u^2 - a^2)\frac{d\log u}{dx} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

shewing that an impressed force is necessary at every place where u is variable and unequal to a.

251. The reason of the change of type which ensues when a wave is left to itself is not difficult to understand. From the ordinary theory we know that an infinitely small disturbance is propagated with a certain velocity a, which velocity is relative to the parts of the medium undisturbed by the wave. Let us consider now the case of a wave so long that the variations of

velocity and density are insensible for a considerable distance along it, and at a place where the velocity (u) is finite let us imagine a small secondary wave to be superposed. The velocity with which the secondary wave is propagated through the medium is a, but on account of the local motion of the medium itself the whole velocity of advance is $a+u$, and depends upon the part of the long wave at which the small wave is placed. What has been said of a secondary wave applies also to the parts of the long wave itself, and thus we see that after a time t the place, where a certain velocity u is to be found, is in advance of its original position by a distance equal, not to at, but to $(a+u)\,t$; or, as we may express it, u is propagated with a velocity $a+u$. In symbolical notation $u=f\{x+(a+u)\,t\}$, where f is an arbitrary function, an equation first obtained by Poisson[1].

From the argument just employed it might appear at first sight that alteration of type was a necessary incident in the progress of a wave, independently of any particular supposition as to the relation between pressure and density, and yet it was proved in § 250 that in the case of one particular law of pressure there would be no alteration of type. We have, however, tacitly assumed in the present section that a is constant, which is tantamount to a restriction to Boyle's law. Under any other law of pressure $\sqrt{\left(\frac{dp}{d\rho}\right)}$ is a function of ρ, and therefore, as we shall see presently, of u. In the case of the law expressed in (5) § 250, the relation between u and ρ for a progressive wave is such that $\sqrt{\left(\frac{dp}{d\rho}\right)}+u$ is constant, as much advance being lost by slower propagation due to augmented density as is gained by superposition of the velocity u.

So far as the constitution of the medium itself is concerned there is nothing to prevent our ascribing arbitrary values to both u and ρ, but in a progressive wave a relation between these two quantities must be satisfied. We know already (§ 245) that this is the case when the disturbance is small, and the following argument will not only shew that such a relation is to be expected in cases where the square of the motion must be retained, but will even define the form of the relation.

[1] Mémoire sur la Théorie du Son. *Journal de l'école polytechnique*, t. VII. p. 319. 1808.

Whatever may be the law of pressure, the velocity of propagation of small disturbances is by § 245 equal to $\sqrt{\left(\frac{dp}{d\rho}\right)}$, and in a positive progressive wave the relation between velocity and condensation is

$$u : s = \sqrt{\left(\frac{dp}{d\rho}\right)} \qquad \text{(1).}$$

If this relation be violated at any point, a wave will emerge, travelling in the negative direction. Let us now picture to ourselves the case of a positive progressive wave in which the changes of velocity and density are very gradual but become important by accumulation, and let us inquire what conditions must be satisfied in order to prevent the formation of a negative wave. It is clear that the answer to the question whether, or not, a negative wave will be generated at any point will depend upon the state of things in the immediate neighbourhood of the point, and not upon the state of things at a distance from it, and will therefore be determined by the criterion applicable to small disturbances. In applying this criterion we are to consider the velocities and condensations, not absolutely, but relatively to those prevailing in the neighbouring parts of the medium, so that the form of (1) proper for the present purpose is

$$du = \sqrt{\left(\frac{dp}{d\rho}\right)} \cdot \frac{d\rho}{\rho} \qquad \text{(2);}$$

whence

$$u = \int \sqrt{\left(\frac{dp}{d\rho}\right)} \cdot \frac{d\rho}{\rho} \qquad \text{(3),}$$

which is the relation between u and ρ necessary for a positive progressive wave. Equation (2) was obtained analytically by Earnshaw[1].

In the case of Boyle's law, $\sqrt{\left(\frac{dp}{d\rho}\right)}$ is constant, and the relation between velocity and density, given first, I believe, by Helmholtz[2], is

$$u = a \log \frac{\rho}{\rho_0} \qquad \text{(4),}$$

if ρ_0 be the density corresponding to $u = 0$.

In this case Poisson's integral allows us to form a definite idea of the change of type accompanying the earlier stages of the

1 *Phil. Trans.* 1859, p. 146.

2 *Fortschritte der Physik*, IV. p. 106. 1852.

progress of the wave, and it finally leads us to a difficulty which has not as yet been surmounted[1]. If we draw a curve to represent the distribution of velocity, taking x for abscissa and u for ordinate, we may find the corresponding curve after the lapse of time t by the following construction. Through any point on the original curve draw a straight line in the positive direction parallel to x, and of length equal to $(a+u)\,t$, or, as we are concerned with the shape of the curve only, equal to ut. The locus of the ends of these lines is the velocity curve after a time t.

But this law of derivation cannot hold good indefinitely. The crests of the velocity curve gain continually on the troughs and must at last overtake them. After this the curve would indicate two values of u for one value of x, ceasing to represent anything that could actually take place. In fact we are not at liberty to push the application of the integral beyond the point at which the velocity becomes discontinuous, or the velocity curve has a vertical tangent. In order to find when this happens let us take two neighbouring points on any part of the curve which slopes downwards in the positive direction, and inquire after what time this part of the curve becomes vertical. If the difference of abscissæ be dx, the hinder point will overtake the forward point in the time $dx \div (-du)$. Thus the motion, as determined by Poisson's equation, becomes discontinuous after a time equal to the reciprocal, taken positively, of the greatest negative value of $\dfrac{du}{dx}$.

For example, let us suppose that

$$u = U \cos \frac{2\pi}{\lambda} \{x - (a+u)\,t\},$$

where U is the greatest initial velocity. When $t=0$, the greatest negative value of $\dfrac{du}{dx}$ is $-\dfrac{2\pi}{\lambda}U$; so that discontinuity will commence at the time $t = \lambda \div 2\pi U$.

When discontinuity sets in, a state of things exists to which the usual differential equations are inapplicable; and the subsequent progress of the motion has not been determined. It is probable, as suggested by Stokes, that some sort of reflection would ensue. In regard to this matter we must be careful to keep

[1] Stokes, "On a difficulty in the Theory of Sound." *Phil. Mag.* Nov. 1848.

purely mathematical questions distinct from physical ones. In practice we have to do with spherical waves, whose divergency may of itself be sufficient to hold in check the tendency to discontinuity. In actual gases too it is certain that before discontinuity could enter, the law of pressure would begin to change its form, and the influence of viscosity could no longer be neglected. But these considerations have nothing to do with the mathematical problem of determining what would happen to waves of finite amplitude in a medium, free from viscosity, whose pressure is under all circumstances exactly proportional to its density; and this problem has not been solved.

It is worthy of remark that, although we may of course conceive a wave of finite disturbance to exist at any moment, there is a limit to the duration of its previous independent existence. By drawing lines in the negative instead of in the positive direction we may trace the history of the velocity curve; and we see that as we push our inquiry further and further into past time the forward slopes become easier and the backward slopes steeper. At a time, equal to the greatest positive value of $\frac{dx}{du}$, antecedent to that at which the curve is first contemplated, the velocity would be discontinuous.

252. The complete integration of the exact equations (4) and (6) § 249 in the case of a progressive wave was first effected by Earnshaw[1]. Finding reason for thinking that in a sound wave the equation

$$\frac{dy}{dt} = F\left(\frac{dy}{dx}\right) \quad\text{..........................}\quad (1)$$

must always be satisfied, he observed that the result of differentiating (1) with respect to t, viz.

$$\frac{d^2y}{dt^2} = \left\{F'\left(\frac{dy}{dx}\right)\right\}^2 \frac{d^2y}{dx^2} \quad\text{......................}\quad (2),$$

can by means of the arbitrary function F be made to coincide with any dynamical equation in which the ratio of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ is expressed in terms of $\frac{dy}{dx}$. The form of the function F being

[1] *Proceedings of the Royal Society*, Jan. 6, 1859. *Phil. Trans.* 1860, p. 133.

thus determined, the solution may be completed by the usual process applicable to such cases[1].

Writing for brevity α in place of $\frac{dy}{dx}$, we have

$$dy = \frac{dy}{dx}dx + \frac{dy}{dt}dt = \alpha\, dx + F(\alpha)\, dt,$$

and the integral is to be found by eliminating α between the equations

$$\left.\begin{aligned} y &= \alpha x + F(\alpha)\, t + \phi(\alpha) \\ 0 &= \quad x + F'(\alpha)\, t + \phi'(\alpha) \end{aligned}\right\} \quad \text{.....................(3)},$$

α being equal to $\rho_0 \div \rho$, and ϕ being an arbitrary function.

If $p = a^2\rho$, the exact equation (6 § 249) is

$$\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2} \quad \text{.......................(4)},$$

by comparison of which with (2) we see that

$$F'(\alpha) = \frac{\pm a}{\alpha} \quad \text{..........................(5)},$$

or on integration

$$F(\alpha) = C \pm a \log \alpha \quad \text{....................(6)},$$

as might also have been inferred from (4) § 251. The constant C vanishes, if $F(\alpha)$, viz. u, vanish when $\alpha = 1$, or $\rho = \rho_0$; otherwise it represents a velocity of the medium as a whole, having nothing to do with the wave as such. For a *positive* progressive wave the lower signs in the ambiguities are to be used. Thus in place of (3), we have

$$\left.\begin{aligned} y &= \alpha x - a \log \alpha\, t + \phi(\alpha) \\ 0 &= \alpha x - a t \quad\;\; + \alpha\phi'(\alpha) \end{aligned}\right\} \quad \text{....................(7)},$$

and

$$u = -a \log \alpha = a \log \frac{\rho}{\rho_0} \quad \text{..........................(8)}.$$

If we subtract the second of equations (7) from the first, we get

$$y - at + at \log \alpha = \phi(\alpha) - \alpha\phi'(\alpha),$$

from which by (8) we see that $y - (a + u)\, t$ is an arbitrary function

[1] Boole's *Differential Equations*, Ch. XIV.

of a, or of u. Conversely therefore u is an arbitrary function of $y-(a+u)t$, and we may write

$$u=f\{y-(a+u)t\} \qquad (9).$$

Equation (9) is Poisson's integral, considered in the preceding section, where the symbol x has the same meaning as here attaches to y.

253. The problem of plane waves of finite amplitude attracted also the attention of Riemann, whose memoir was communicated to the Royal Society of Göttingen on the 28th of November, 1859[1]. Riemann's investigation is founded on the general hydrodynamical equations investigated in §§ 237, 238, and is not restricted to any particular law of pressure. In order, however, not unduly to extend the discussion of this part of our subject, already perhaps treated at greater length than its physical importance would warrant, we shall here confine ourselves to the case of Boyle's law of pressure.

Applying equations (1), (2) of § 237 and (1) of § 238 to the circumstances of the present problem, we get

$$\frac{du}{dt}+u\frac{du}{dx}=-a^2\frac{d\log\rho}{dx} \qquad (1),$$

$$\frac{d\log\rho}{dt}+u\frac{d\log\rho}{dx}=-\frac{du}{dx} \qquad (2).$$

If we multiply (2) by $\pm a$, and afterwards add it to (1), we obtain

$$\frac{dP}{dt}=-(u+a)\frac{dP}{dx}, \qquad \frac{dQ}{dt}=-(u-a)\frac{dQ}{dx} \qquad (3),$$

where $$P=a\log\rho+u, \qquad Q=a\log\rho-u \qquad (4).$$

Thus

$$dP=\frac{dP}{dx}\{dx-(u+a)\,dt\} \qquad (5),$$

$$dQ=\frac{dQ}{dx}\{dx-(u-a)\,dt\} \qquad (6).$$

[1] Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. Göttingen, *Abhandlungen*, t. VIII. 1860. See also an excellent abstract in the *Fortschritte der Physik*, XV. p. 123.

These equations are more general than Poisson's and Earnshaw's in that they are not limited to the case of a single positive, or negative, progressive wave. From (5) we learn that whatever may be the value of P corresponding to the point x and the time t, the same value of P corresponds to the point $x+(u+a)\,dt$ at the time $t+dt$; and in the same way from (6) we see that Q remains unchanged when x and t acquire the increments $(u-a)\,dt$ and dt respectively. If P and Q be given at a certain instant of time as functions of x, and the representative curves be drawn, we may deduce the corresponding value of u by (4), and thus, as in § 251, construct the curves representing the values of P and Q after the small interval of time dt, from which the new values of u and ρ in their turn become known, and the process can be repeated.

The element of the fluid, to which the values of P and Q at any moment belong, is itself moving with the velocity u, so that the velocities of P and Q relatively to the element are numerically the same, and equal to a, that of P being in the positive direction and that of Q in the negative direction.

We are now in a position to trace the consequences of an initial disturbance which is confined to a finite portion of the medium, *e.g.* between $x=\alpha$ and $x=\beta$, outside which the medium is at rest and at its normal density, so that the values of P and Q are $a\log\rho_0$. Each value of P propagates itself in turn to the elements of fluid which lie in front of it, and each value of Q to those that lie behind it. The hinder limit of the region in which P is variable, viz. the place where P first attains the constant value $a\log\rho_0$, comes into contact first with the variable values of Q, and moves accordingly with a variable[1] velocity. At a definite time, requiring for its determination a solution of the differential equations, the hinder (left hand) limit of the region through which P varies, meets the hinder (right hand) limit of the region through which Q varies, after which the two regions separate themselves, and include between them a portion of fluid in its equilibrium condition, as appears from the fact that the values of P and Q are both $a\log\rho_0$. In the positive wave Q has the constant value $a\log\rho_0$, so that $u=a\log\dfrac{\rho}{\rho_0}$, as in (4) § 251; in the negative wave

[1] At this point an error seems to have crept into Riemann's work, which is corrected in the abstract of the *Fortschritte der Physik*.

P has the same constant value, giving as the relation between u and ρ, $u = -a \log \frac{\rho}{\rho_0}$. Since in each progressive wave, when isolated, a law prevails connecting the quantities u and ρ, we see that in the positive wave du vanishes with dP, and in the negative wave du vanishes with dQ. Thus from (5) we learn that in a positive progressive wave du vanishes, if the increments of x and t be such as to satisfy the equation $dx - (u + a)\,dt = 0$, from which Poisson's integral immediately follows.

It would lead us too far to follow out the analytical development of Riemann's method, for which the reader must be referred to the original memoir; but it would be improper to pass over in silence an error on the subject of discontinuous motion into which Riemann and other writers have fallen. It has been held that a state of motion is possible in which the fluid is divided into two parts by a surface of discontinuity propagating itself with constant velocity, all the fluid on one side of the surface of discontinuity being in one uniform condition as to density and velocity, and on the other side in a second uniform condition in the same respects. Now, if this motion were possible, a motion of the same kind in which the surface of discontinuity is at rest would also be possible, as we may see by supposing a velocity equal and opposite to that with which the surface of discontinuity at first moves, to be impressed upon the whole mass of fluid. In order to find the relations that must subsist between the velocity and density on the one side (u_1, ρ_1) and the velocity and density on the other side (u_2, ρ_2), we notice in the first place that by the principle of conservation of matter $\rho_2 u_2 = \rho_1 u_1$. Again, if we consider the momentum of a slice bounded by parallel planes and including the surface of discontinuity, we see that the momentum leaving the slice in the unit of time is for each unit of area $(\rho_2 u_2 = \rho_1 u_1)\, u_2$, while the momentum entering it is $\rho_1 u_1^2$. The difference of momentum must be balanced by the pressures acting at the boundaries of the slice, so that

$$\rho_1 u_1 (u_2 - u_1) = p_1 - p_2 = a^2 (\rho_1 - \rho_2),$$

whence

$$u_1 = a\sqrt{\left(\frac{\rho_2}{\rho_1}\right)}, \qquad u_2 = a\sqrt{\left(\frac{\rho_1}{\rho_2}\right)} \quad\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

The motion thus determined is, however, not possible; it satisfies

indeed the conditions of mass and momentum, but it violates the condition of energy (§ 244) expressed by the equation

$$\tfrac{1}{2} u_2^2 - \tfrac{1}{2} u_1^2 = a^2 \log \rho_1 - a^2 \log \rho_2 \text{.............} (8).$$

This argument has been already given in another form in § 250, which would alone justify us in rejecting the assumed motion, since it appears that no steady motion is possible except under the law of density there determined. From equation (8) of that section we can find what impressed forces would be necessary to maintain the motion defined by (7). It appears that the force X, though confined to the place of discontinuity, is made up of two parts of opposite signs, since by (7) u passes *through* the value a. The whole moving force, viz. $\int X\rho\, dx$, vanishes, and this explains how it is that the condition relating to momentum is satisfied by (7), though the force X be ignored altogether.

254. The exact experimental determination of the velocity of sound is a matter of greater difficulty than might have been expected. Observations in the open air are liable to errors from the effects of wind, and from uncertainty with respect to the exact condition of the atmosphere as to temperature and dryness. On the other hand when sound is propagated through air contained in pipes, disturbance arises from friction and from transfer of heat; and, although no great errors from these sources are to be feared in the case of tubes of considerable diameter, such as some of those employed by Regnault, it is difficult to feel sure that the ideal plane waves of theory are nearly enough realized.

The following Table[1] contains a list of the principal experimental determinations which have been made hitherto.

Names of Observers.	Velocity of Sound at 0° Cent. in Metres.
Académie des Sciences (1738)	332
Benzenberg (1811)	333·7 332·3
Goldingham (1821)	331·1
Bureau des Longitudes (1822)	330·6
Moll and van Beek	332·2

[1] Bosanquet, *Phil. Mag.* April, 1877.

Names of Observers.	Velocity of Sound at 0° Cent. in Metres.
Stampfer and Myrback	332·4
Bravais and Martins (1844)	332·4
Wertheim	331·6
Stone (1871)	332·4
Le Roux	330·7
Regnault	330·7

In Stone's experiments[1] the course over which the sound was timed commenced at a distance of 640 feet from the source, so that any errors arising from excessive disturbance were to a great extent avoided.

A method has been proposed by Bosscha[2] for determining the velocity of sound without the use of great distances. It depends upon the precision with which the ear is able to decide whether short ticks are simultaneous, or not. In König's[3] form of the experiment, two small electro-magnetic counters are controlled by a fork-interrupter (§ 64), whose period is one-tenth of a second, and give synchronous ticks of the same period. When the counters are close together the audible ticks coincide, but as one counter is gradually removed from the ear, the two series of ticks fall asunder. When the difference of distances is about 34 metres, coincidence again takes place, proving that 34 metres is about the distance traversed by sound in a tenth part of a second.

[1] *Phil. Trans.* 1872, p. 1. [2] *Pogg. Ann.* XCII. 486. 1854.
[3] *Pogg. Ann.* CXVIII. 610. 1863.

CHAPTER XII.

VIBRATIONS IN TUBES.

255. We have already (§ 245) considered the solution of our fundamental equation, when the velocity-potential, in an unlimited fluid, is a function of one space co-ordinate only. In the absence of friction no change would be caused by the introduction of any number of fixed cylindrical surfaces, whose generating lines are parallel to the co-ordinate in question; for even when the surfaces are absent the fluid has no tendency to move across them. If one of the cylindrical surfaces be closed (in respect to its transverse section), we have the important problem of the axial motion of air within a cylindrical pipe, which, when once the mechanical conditions at the ends are given, is independent of anything that may happen outside the pipe.

Considering a simple harmonic vibration, we know (§ 245) that, if ϕ varies as e^{int},

$$\frac{d^2\phi}{dx^2}+\kappa^2\phi=0 \qquad\qquad (1),$$

where

$$\kappa=\frac{2\pi}{\lambda}=\frac{n}{a} \qquad\qquad (2).$$

The solution may be written in two forms—

$$\left.\begin{aligned}\phi&=(A\cos\kappa x+B\sin\kappa x)\,e^{int}\\ \phi&=(A\,e^{i\kappa x}+B\,e^{-i\kappa x})\,e^{int}\end{aligned}\right\} \qquad\qquad (3),$$

of which finally only the real parts will be retained. The first form will be most convenient when the vibration is stationary, or

nearly so, and the second when the motion reduces itself to a positive, or negative, progressive undulation. The constants A and B in the symbolical solution may be complex, and thus the final expression in terms of real quantities will involve *four* arbitrary constants. If we wish to use real quantities throughout, we must take

$$\phi = (A \cos \kappa x + B \sin \kappa x) \cos nt$$
$$+ (C \cos \kappa x + D \sin \kappa x) \sin nt \quad\ldots\ldots\ldots\ldots(4),$$

but the analytical work would generally be longer. When no ambiguity can arise, we shall sometimes for the sake of brevity drop, or restore, the factor involving the time without express mention. Equations such as (1) are of course equally true whether the factor be understood or not.

Taking the first form in (3), we have

$$\left.\begin{aligned} \phi &= A \cos \kappa x + B \sin \kappa x \\ \frac{d\phi}{dx} &= -\kappa A \sin \kappa x + \kappa B \cos \kappa x \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(5).$$

If there be any point at which either ϕ or $\frac{d\phi}{dx}$ is permanently zero, the ratio $A : B$ must be real, and then the vibration is *stationary*, that is, the same in phase at all points simultaneously.

Let us suppose that there is a node at the origin. Then when $x = 0$, $\frac{d\phi}{dx}$ vanishes, the condition of which is $B = 0$. Thus

$$\phi = A \cos \kappa x \, e^{int}; \quad \frac{d\phi}{dx} = -\kappa A \sin \kappa x \, e^{int} \quad\ldots\ldots\ldots\ldots(6),$$

from which, if we substitute $Pe^{i\theta}$ for A, and throw away the imaginary part,

$$\left.\begin{aligned} \phi &= P \cos \kappa x \cos (nt + \theta) \\ \frac{d\phi}{dx} &= -\kappa P \sin \kappa x \cos (nt + \theta) \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(7).$$

From these equations we learn that $\frac{d\phi}{dx}$ vanishes wherever $\sin \kappa x = 0$; that is, that besides the origin there are nodes at the points $x = \frac{1}{2} m\lambda$, m being any positive or negative integer. At any of these places infinitely thin rigid plane barriers normal to x might be stretched across the tube without in any way alter-

ing the motion. Midway between each pair of consecutive nodes there is a *loop*, or place of no pressure variation, since $\delta p = -\rho\dot{\phi}$ (6) § 244. At any of these loops a communication with the external atmosphere might be opened, without causing any disturbance of the motion from air passing in or out. The loops are the places of maximum velocity, and the nodes those of maximum pressure variation. At intervals of λ everything is exactly repeated.

If there be a node at $x = l$, as well as at the origin, $\sin \kappa l = 0$, or $\lambda = 2l \div m$, where m is a positive integer. The gravest tone which can be sounded by air contained in a doubly closed pipe of length l is therefore that which has a wave-length equal to $2l$. This statement, it will be observed, holds good whatever be the gas with which the pipe is filled; but the frequency, or the place of the tone in the musical scale, depends also on the nature of the particular gas. The periodic time is given by

$$\tau = \frac{\lambda}{a} = \frac{2l}{a} \qquad\qquad (8).$$

The other tones possible for a doubly closed pipe have periods which are submultiples of that of the gravest tone, and the whole system forms a harmonic scale.

Let us now suppose, without stopping for the moment to inquire how such a condition of things can be secured, that there is a loop instead of a node at the point $x = l$. Equation (6) gives $\cos \kappa l = 0$, whence $\lambda = 4l \div (2m + 1)$, where m is zero or a positive integer. In this case the gravest tone has a wave-length equal to four times the length of the pipe reckoned from the node to the loop, and the other tones form with it a harmonic scale, from which, however, all the members of even order are missing.

256. By means of a rigid barrier there is no difficulty in securing a node at any desired point of a tube, but the condition for a loop, i.e. that under no circumstances shall the pressure vary, can only be realized approximately. In most cases the variation of pressure at any point of a pipe may be made small by allowing a free communication with the external air. Thus Euler and Lagrange assumed constancy of pressure as the condition to be satisfied at the end of an open pipe. We shall afterwards return to the problem of the open pipe, and investigate by a rigorous

process the conditions to be satisfied at the end. For our immediate purpose it will be sufficient to know, what is indeed tolerably obvious, that the open end of a pipe may be treated as a loop, if the diameter of the pipe be neglected in comparison with the wave-length, provided the external pressure in the neighbourhood of the open end be not itself variable from some cause independent of the motion within the pipe. When there is an independent source of sound, the pressure at the end of the pipe is the same as it would be in the same place, if the pipe were away. The impediment to securing the fulfilment of the condition for a loop at any desired point lies in the inertia of the machinery required to sustain the pressure. For theoretical purposes we may overlook this difficulty, and imagine a massless piston backed by a compressed spring also without mass. The assumption of a loop at an open end of a pipe, is tantamount to neglecting the inertia of the outside air.

We have seen that, if a node exist at any point of a pipe, there must be a series, ranged at equal intervals $\frac{1}{2}\lambda$, that midway between each pair of consecutive nodes there must be a loop, and that the whole vibration must be stationary. The same conclusion follows if there be at any point a loop; but it may perfectly well happen that there are neither nodes nor loops, as for example in the case when the motion reduces to a positive or negative progressive wave. In stationary vibration there is no transference of energy along the tube in either direction, for energy cannot pass a node or a loop.

257. The relations between the lengths of an open or closed pipe and the wave-lengths of the included column of air may also be investigated by following the motion of a *pulse*, by which is understood a wave confined within narrow limits and composed of uniformly condensed or rarefied fluid. In looking at the matter from this point of view it is necessary to take into account carefully the circumstances under which the various reflections take place. Let us first suppose that a condensed pulse travels in the positive direction towards a barrier fixed across the tube. Since the energy contained in the wave cannot escape from the tube, there must be a reflected wave, and that this reflected wave is also a wave of condensation appears from the fact that there is no loss of fluid. The same conclusion may be arrived at in another way. The effect of the barrier may be imitated by the introduc-

tion of a similar and equidistant wave of condensation moving in the negative direction. Since the two waves are both condensed and are propagated in contrary directions, the velocities of the fluid composing them are equal and opposite, and therefore neutralise one another when the waves are superposed.

If the progress of the negative reflected wave be interrupted by a second barrier, a similar reflection takes place, and the wave, still remaining condensed, regains its positive character. When a distance has been travelled equal to twice the length of the pipe, the original state of things is completely restored, and the same cycle of events repeats itself indefinitely. We learn therefore that the period within a doubly closed pipe is the time occupied by a pulse in travelling twice the length of the pipe.

The case of an open end is somewhat different. The supplementary negative wave necessary to imitate the effect of the open end must evidently be a wave of rarefaction capable of neutralizing the positive pressure of the condensed primary wave, and thus in the act of reflection a wave changes its character from condensed to rarefied, or from rarefied to condensed. Another way of considering the matter is to observe that in a positive condensed pulse the momentum of the motion is forwards, and in the absence of the necessary forces cannot be changed by the reflection. But forward motion in the reflected negative wave is indissolubly connected with the rarefied condition.

When both ends of a tube are open, a pulse travelling backwards and forwards within it is completely restored to its original state after traversing twice the length of the tube, suffering in the process two reflections, and thus the relation between length and period is the same as in the case of a tube, whose ends are both closed; but when one end of a tube is open and the other closed, a double passage is not sufficient to close the cycle of changes. The original condensed or rarefied character cannot be recovered until after two reflections from the open end, and accordingly in the case contemplated the period is the time required by the pulse to travel over *four* times the length of the pipe.

258. After the full discussion of the corresponding problems in the chapter on Strings, it will not be necessary to say much on the compound vibrations of columns of air. As a simple example we may take the case of a pipe open at one end and closed at the

other, which is suddenly brought to rest at the time $=0$, after being for some time in motion with a uniform velocity parallel to its length. The initial state of the contained air is then one of uniform velocity u_0 parallel to x, and of freedom from compression and rarefaction. If we suppose that the origin is at the closed end, the general solution is by (7) § 255,

$$\phi = (A_1 \cos n_1 t + B_1 \sin n_1 t) \cos \kappa_1 x$$
$$+ (A_2 \cos n_2 t + B_2 \sin n_2 t) \cos \kappa_2 x$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $\kappa_r = \frac{2r-1}{2}\frac{\pi}{l}$, $n_r = a\kappa_r$, and A_1, B_1, A_2, B_2 .. are arbitrary constants.

Since $\dot\phi$ is to be zero initially for all values of x, the coefficients B must vanish; the coefficients A are to be determined by the condition that for all values of x between 0 and l,

$$\Sigma\, \kappa_r A_r \sin \kappa_r x = -u_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where the summation extends to all integral values of r from 1 to ∞. The determination of the coefficients A from (2) is effected in the usual way. Multiplying by $\sin \kappa_r x\, dx$, and integrating from 0 to l, we get

$$\tfrac{1}{2} l\, \kappa_r A_r = -\frac{u_0}{\kappa_r},$$

or

$$A_r = -\frac{2u_0}{\kappa_r^2 l} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The complete solution is therefore

$$\phi = -\frac{2u_0}{l} \Sigma_{r=1}^{r=\infty} \frac{\cos \kappa_r x}{\kappa_r^2} \cos n_r t \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

259. In the case of a tube stopped at the origin and open at $x=l$, let $\phi = \cos nt$ be the value of the potential at the open end due to an external source of sound. Determining P and θ in equation (7) § 255, we find

$$\phi = \frac{\cos \kappa x}{\cos \kappa l} \cos nt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

It appears that the vibration within the tube is a minimum, when $\cos \kappa l = \pm 1$, that is when l is a multiple of $\frac{1}{2}\lambda$, in which case

there is a node at $x=l$. When l is an odd multiple of $\frac{1}{4}\lambda$, $\cos \kappa l$ vanishes, and then according to (1) the motion would become infinite. In this case the supposition that the pressure at the open end is independent of what happens within the tube breaks down; and we can only infer that the vibration is very large, in consequence of the isochronism. Since there is a node at $x=0$, there must be a loop when x is an odd multiple of $\frac{1}{4}\lambda$, and we conclude that in the case of isochronism the variation of pressure at the open end of the tube due to the external cause is exactly neutralised by the variation of pressure due to the motion within the tube itself. If there were really at the open end a variation of pressure on the whole, the motion must increase without limit in the absence of dissipative forces.

If we suppose that the origin is a loop instead of a node, the solution is

$$\phi = \frac{\sin \kappa x}{\sin \kappa l} \cos nt \quad\text{.........................}(2),$$

where $\phi = \cos nt$ is the given value of ϕ at the open end $x=l$. In this case the expression becomes infinite, when $\kappa l = m\pi$, or $l = \frac{1}{2} m\lambda$.

We will next consider the case of a tube, whose ends are both open and exposed to disturbances of the same period, making ϕ equal to He^{int}, Ke^{int} respectively. Unless the disturbances at the ends are in the same phase, one at least of the coefficients H, K must be complex.

Taking the first form in (3) § 255, we have as the general expression for ϕ

$$\phi = e^{int}(A \cos \kappa x + B \sin \kappa x).$$

If we take the origin in the middle of the tube, and assume that the values He^{int}, Ke^{int} correspond respectively to $x=l$, $x=-l$, we get to determine A and B,

$$H = A \cos \kappa l + B \sin \kappa l,$$

$$K = A \cos \kappa l - B \sin \kappa l,$$

whence

$$A = \frac{H+K}{2 \cos \kappa l}, \qquad B = \frac{H-K}{2 \sin \kappa l} \quad\text{..................}(3),$$

giving

$$\phi = e^{int}\frac{H \sin \kappa (l+x) + K \sin \kappa (l-x)}{\sin 2\kappa l} \quad\text{..........}(4).$$

This result might also be deduced from (2), if we consider that the required motion arises from the superposition of the motion, which is due to the disturbance He^{int} calculated on the hypothesis that the other end $x=-l$ is a loop, on the motion, which is due to Ke^{int} on the hypothesis that the end $x=l$ is a loop.

The vibration expressed by (4) cannot be *stationary*, unless the ratio $H:K$ be real, that is unless the disturbances at the ends be in similar, or in opposite, phases. Hence, except in the cases reserved, there is no loop anywhere, and therefore no place at which a branch tube can be connected along which sound will not be propagated[1].

At the middle of the tube, for which $x=0$,

$$\phi = \frac{H+K}{2\cos\kappa l}e^{int}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

shewing that the variation of pressure (proportional to $\dot{\phi}$) vanishes if $H+K=0$, that is, if the disturbances at the ends be equal and in *opposite* phases. Unless this condition be satisfied, the expression becomes infinite, when $2l=\frac{1}{2}(2m+1)\lambda$.

At a point distant $\frac{1}{4}\lambda$ from the middle of the tube the expression for ϕ is

$$\phi = \frac{H-K}{2\sin\kappa l}e^{int}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

vanishing when $H=K$, that is, when the disturbances at the ends are equal and in the *same* phase. In general ϕ becomes infinite, when $\sin\kappa l=0$, or $2l=m\lambda$.

If at one end of an unlimited tube there be a variation of pressure due to an external source, a train of progressive waves will be propagated inwards from that end. Thus, if the length along the tube measured from the open end be y, the velocity-potential is expressed by $\phi=\cos\left(nt-n\dfrac{y}{a}\right)$, corresponding to

[1] An arrangement of this kind has been proposed by Prof. Mayer (*Phil. Mag.* XLV. p. 90, 1873) for comparing the intensities of sources of sound of the same pitch. Each end of the tube is exposed to the action of one of the sources to be compared, and the distances are adjusted until the amplitudes of the vibrations denoted by H and K are equal. The branch tube is led to a manometric capsule (§ 262), and the method assumes that by varying the point of junction the disturbance of the flame can be stopped. From the discussion in the text it appears that this assumption is not theoretically correct.

$\phi = \cos nt$ at $y = 0$; so that, if the cause of the disturbance within the tube be the passage of a train of progressive waves across the open end, the intensity within the tube will be the same as in the space outside. It must not be forgotten that the diameter of the tube is supposed to be infinitely small in comparison with the length of a wave.

Let us next suppose that the source of the motion is within the tube itself, due for example to the inexorable motion of a piston at the origin[1]. The constants in (5) § 255 are to be determined by the conditions that when $x = 0$, $\dfrac{d\phi}{dx} = \cos nt$ (say), and that, when $x = l$, $\phi = 0$. Thus $\kappa A = -\tan \kappa l$, $\kappa B = 1$, and the expression for ϕ is

$$\phi = \frac{\sin \kappa (x - l)}{\kappa \cos \kappa l} \qquad\qquad (7).$$

The motion is a minimum, when $\cos \kappa l = \pm 1$, that is, when the length of the tube is a multiple of $\frac{1}{2}\lambda$.

When l is an odd multiple of $\frac{1}{4}\lambda$, the place occupied by the piston would be a node, if the open end were really a loop, but in this case the solution fails. The escape of energy from the tube prevents the energy from accumulating beyond a certain point; but no account can be taken of this so long as the open end is treated rigorously as a loop. We shall resume the question of resonance after we have considered in greater detail the theory of the open end, when we shall be able to deal with it more satisfactorily.

In like manner if the point $x = l$ be a node, instead of a loop, the expression for ϕ is

$$\phi = \frac{\cos \kappa (l - x)}{\kappa \sin \kappa l} \qquad\qquad (8);$$

and thus the motion is a minimum when l is an odd multiple of $\frac{1}{4}\lambda$, in which case the origin is a loop. When l is an even multiple of $\frac{1}{4}\lambda$, the origin should be a node, which is forbidden by the conditions of the question. In this case according to (8) the motion becomes infinite, which means that in the absence of dissipative forces the vibration would increase without limit.

[1] These problems are considered by Poisson, *Mém. de l'Institut*, t. II. p. 305.

260. The experimental investigation of aerial waves within pipes has been effected with considerable success by M. Kundt[1]. To generate waves is easy enough; but it is not so easy to invent a method by which they can be effectually examined. M. Kundt discovered that the nodes of stationary waves can be made evident by dust. A little fine sand or lycopodium seed, shaken over the interior of a glass tube containing a vibrating column of air, disposes itself in recurring patterns, by means of which it is easy to determine the positions of the nodes and to measure the intervals between them. In Kundt's experiments the origin of the sound was in the longitudinal vibration of a glass tube called the sounding-tube, and the dust-figures were formed in a second and larger tube, called the wave-tube, the latter being provided with a moveable stopper for the purpose of adjusting its length. The other end of the wave-tube was fitted with a cork through which the sounding-tube passed half way. By suitable friction the sounding-tube was caused to vibrate in its gravest mode, so that the central point was nodal, and its interior extremity (closed with a cork) excited aerial vibrations in the wave-tube. By means of the stopper the length of the column of air could be adjusted so as to make the vibrations as vigorous as possible, which happens when the interval between the stopper and the end of the sounding-tube is a multiple of half the wave-length of the sound.

With this apparatus Kundt was able to compare the wave-lengths of the same sound in various gases, from which the relative velocities of propagation are at once deducible, but the results were not entirely satisfactory. It was found that the intervals of recurrence of the dust-patterns were not strictly equal, and, what was worse, that the pitch of the sound was not constant from one experiment to another. These defects were traced to a communication of motion to the wave-tube through the cork, by which the dust-figures were disturbed, and the pitch made irregular in consequence of unavoidable variations in the mounting of the apparatus. To obviate them, Kundt replaced the cork, which formed too stiff a connection between the tubes, by layers of sheet indiarubber tied round with silk, obtaining in this way a flexible and perfectly air-tight joint; and in order to avoid any risk of the comparison of wave-lengths being vitiated by an alteration of pitch,

[1] *Pogg. Ann.* t. CXXXV. p. 337. 1868.

the apparatus was modified so as to make it possible to excite the two systems of dust-figures simultaneously and in response to the same sound. A collateral advantage of the new method consisted in the elimination of temperature-corrections.

In the improved "Double Apparatus" the sounding-tube was caused to vibrate *in its second mode* by friction applied near the middle; and thus the nodes were formed at the points distant from the ends by one-fourth of the length of the tube. At each of these points connection was made with an independent wave-tube, provided with an adjustable stopper, and with branch tubes and stop-cocks suitable for admitting the various gases to be experimented upon. It is evident that dust-figures formed in the two tubes correspond rigorously to the same pitch, and that therefore a comparison of the intervals of recurrence leads to a correct determination of the velocities of propagation, under the circumstances of the experiment, for the two gases with which the tubes are filled.

The results at which Kundt arrived were as follows:—

(*a*) The velocity of sound in a tube diminishes with the diameter. Above a certain diameter, however, the change is not perceptible.

(*b*) The diminution of velocity increases with the wave-length of the tone employed.

(*c*) Powder, scattered in a tube, diminishes the velocity of sound in narrow tubes, but in wide ones is without effect.

(*d*) In narrow tubes the effect of powder increases, when it is very finely divided, and is strongly agitated in consequence.

(*e*) Roughening the interior of a narrow tube, or increasing its surface, diminishes the velocity.

(*f*) In wide tubes these changes of velocity are of no importance, so that the method may be used in spite of them for exact determinations.

(*g*) The influence of the intensity of sound on the velocity cannot be proved.

(*h*) With the exception of the first, the wave-lengths of a tone as shewn by dust are not affected by the mode of excitation.

(*i*) In wide tubes the velocity is independent of pressure, but in small tubes the velocity increases with the pressure.

(*j*) All the observed changes in the velocity were due to friction, and especially to exchange of heat between the air and the sides of the tube.

(*k*) The velocity of sound at 100° agrees exactly with that given by theory[1].

We shall return to the question of the propagation of sound in narrow tubes as affected by the causes mentioned above (*j*), and shall then investigate the formulæ given by Helmholtz and Kirchhoff.

261. In the experiments described in the preceding section the aerial vibrations are *forced*, the pitch being determined by the external source, and not (in any appreciable degree) by the length of the column of air. Indeed, strictly speaking, all sustained vibrations are forced, as it is not in the power of free vibrations to maintain themselves, except in the ideal case when there is absolutely no friction. Nevertheless there is an important practical distinction between the vibrations of a column of air as excited by a longitudinally vibrating rod or by a tuning-fork, and such vibrations as those of the organ-pipe or chemical harmonicon. In the latter cases the pitch of the sound depends principally on the length of the aerial column, the function of the wind or of the flame[2] being merely to restore the energy lost by friction and by communication to the external air. The air in an organ-pipe is to be considered as a column swinging almost freely, the lower end, across which the wind sweeps, being treated roughly as open, and the upper end as closed, or open, as the case may be. Thus the wave-length of the principal tone of a stopped pipe is four times the length of the pipe; and, except at the extremities, there is neither node nor loop. The overtones of the pipe are the *odd* harmonics, twelfth, higher third, &c., corresponding to the various subdivisions of the column of air. In the case of the twelfth, for example, there is a node at the point of trisection nearest to the

[1] From some expressions in the memoir already cited, from which the notice in the text is principally derived, M. Kundt appears to have contemplated a continuation of his investigations; but I am unable to find any later publication on the subject.

[2] The subject of sensitive flames with and without pipes is treated in considerable detail by Prof. Tyndall in his work on Sound; but the mechanics of this class of phenomena is still very imperfectly understood. We shall return to it in a subsequent chapter.

open end, and a loop at the other point of trisection midway between the first and the stopped end of the pipe.

In the case of the open organ-pipe both ends are loops, and there must be at least one internal node. The wave-length of the principal tone is twice the length of the pipe, which is divided into two similar parts by a node in the middle. From this we see the foundation of the ordinary rule that the pitch of an open pipe is the same as that of a stopped pipe of half its length. For reasons to be more fully explained in a subsequent chapter, connected with our present imperfect treatment of the open end, the rule is only approximately correct. The open pipe, differing in this respect from the stopped pipe, is capable of sounding the whole series of tones forming the harmonic scale founded upon its principal tone. In the case of the octave there is a loop at the centre of the pipe and nodes at the points midway between the centre and the extremities.

Since the frequency of the vibration in a pipe is proportional to the velocity of propagation of sound in the gas with which the pipe is filled, the comparison of the pitches of the notes obtained from the same pipe in different gases is an obvious method of determining the velocity of propagation, in cases where the impossibility of obtaining a sufficiently long column of the gas precludes the use of the direct method. In this application Chladni with his usual sagacity led the way. The subject was resumed at a later date by Dulong[1] and by Wertheim[2], who obtained fairly satisfactory results.

262. The condition of the air in the interior of an organ-pipe was investigated experimentally by Savart[3], who lowered into the pipe a small stretched membrane on which a little sand was scattered. In the neighbourhood of a node the sand remained sensibly undisturbed, but, as a loop was approached, it danced with more and more vigour. But by far the most striking form of the experiment is that invented by König. In this method the vibration is indicated by a small gas flame, fed through a tube which is in communication with a cavity called a manometric capsule.

[1] Recherches sur les chaleurs spécifiques des fluides élastiques. *Ann. d. Chim.*, t. XLI. p. 113.

[2] *Ann. de Chim.*, 3ième série, t. XXII. p. 434.

[3] *Ann. de Chim.*, t. XXIV. p. 56. 1823.

This cavity is bounded on one side by a membrane on which the vibrating air acts. As the membrane vibrates, rendering the capacity of the capsule variable, the supply of gas becomes unsteady and the flame intermittent. The period is of course too small for the intermittence to manifest itself as such when the flame is looked at steadily. By shaking the head, or with the aid of a moveable mirror, the resolution into more or less detached images may be effected; but even without resolution the altered character of the flame is evident from its general appearance. In the application to organ-pipes, one or more capsules are mounted on a pipe in such a manner that the membranes are in contact with the vibrating column of air; and the difference in the flame is very marked, according as the associated capsule is situated at a node or at a loop.

263. Hitherto we have supposed the pipe to be straight, but it will readily be anticipated that, when the cross section is small and does not vary in area, straightness is not a matter of importance. Conceive a curved axis of x running along the middle of the pipe, and let the constant section perpendicular to this axis be S. When the greatest diameter of S is very small in comparison with the wave-length of the sound, the velocity-potential ϕ becomes nearly invariable over the section; applying Green's theorem to the space bounded by the interior of the pipe and by two cross sections, we get

$$\iiint \nabla^2\phi \, dV = S \,.\, \Delta\left(\frac{d\phi}{dx}\right).$$

Now by the general equation of motion

$$\iiint \nabla^2\phi \, dV = \frac{1}{a^2}\iiint \ddot{\phi}\, dV = \frac{1}{a^2}\frac{d^2}{dt^2}\iiint \phi \, dV = \frac{S}{a^2}\frac{d^2}{dt^2}\int \phi \, dx,$$

and in the limit, when the distance between the sections is made to vanish,

$$\int \phi \, dx = \phi \, dx, \qquad \Delta\left(\frac{d\phi}{dx}\right) = \frac{d^2\phi}{dx^2}\,dx;$$

so that

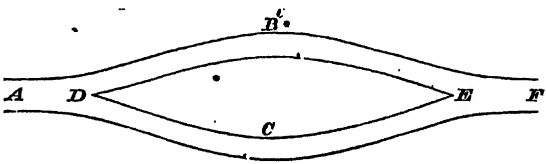
$$\frac{d^2\phi}{dt^2} = a^2\frac{d^2\phi}{dx^2} \quad\text{..............................(1)},$$

shewing that ϕ depends upon x in the same way as if the pipe were straight. By means of equation (1) the vibrations of air in

curved pipes of uniform section may be easily investigated, and the results are the rigorous consequences of our fundamental equations (which take no account of friction), when the section is supposed to be infinitely small. In the case of thin tubes such as would be used in experiment, they suffice at any rate to give a very good representation of what actually happens.

264. We now pass on to the consideration of certain cases of connected tubes. In the accompanying figure AB represents a thin pipe, which divides at D into two branches DB, DC. At E the branches reunite and form a single tube EF. The sections of the single tubes and of the branches are assumed to be uniform as well as very small.

Fig. 55.



In the first instance let us suppose that a positive wave of arbitrary type is advancing in A. On its arrival at the fork D, it will give rise to positive waves in B and C, and, unless a certain condition be satisfied, to a negative reflected wave in A. Let the potential of the positive waves be denoted by f_A, f_B, f_C, f being in each case a function of $x - at$; and let the reflected wave be $F(x+at)$. Then the conditions to be satisfied at D are first that the pressures shall be the same for the three pipes, and secondly that the whole velocity of the fluid in A shall be equal to the sum of the whole velocities of the fluid in B and C. Thus, using A, B, C to denote the areas of the sections, we have, § 244,

$$\left.\begin{aligned} f_A' - F' &= f_B' = f_C' \\ A(f_A' + F') &= Bf_B' + Cf_C' \end{aligned}\right\} \quad\text{.........................(1)};$$

whence

$$F' = \frac{B+C-A}{B+C+A} f_A' \quad\text{.............................(2)},$$

$$f_B' = f_C' = \frac{2A}{B+C+A} f_A' \quad\text{.........................(3)[1]}.$$

[1] These formulæ, as applied to determine the reflected and refracted waves at the junction of two tubes of sections $B+C$, and A respectively, are given by

It appears that f_B and f_C are always the same. There is no reflection, if

$$B+C=A \quad\text{..............................(4)},$$

that is, if the combined sections of the branches be equal to the section of the trunk; and, when this condition is satisfied,

$$f_B=f_C=f_A \quad\text{..............................(5)}.$$

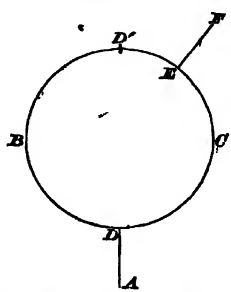
The wave then advances in B and C exactly as it would have done in A, had there been no break. If the lengths of the branches between D and E be equal, and the section of F be equal to that of A, the waves on arrival at E combine into a wave propagated along F, and again there is no reflection. The division of the tube has thus been absolutely without effect; and since the same would be true for a negative wave passing from F to A, we may conclude generally that a tube may be divided into two, or more, branches, all of the same length, without in any way influencing the law of aërial vibration, provided that the whole section remain constant. If the lengths of the branches from D to E be unequal, the result is different. Besides the positive wave in F, there will be in general negative reflected waves in B and C. The most interesting case is when the wave is of harmonic type and one of the branches is longer than the other by a multiple of $\frac{1}{2}\lambda$. If the difference be an *even* multiple of $\frac{1}{2}\lambda$, the result will be the same as if the branches were of equal length, and no reflection will ensue. But suppose that, while B and C are equal in section, one of them is longer than the other by an *odd* multiple of $\frac{1}{2}\lambda$. Since the waves arrive at E in opposite phases, it follows from symmetry that the positive wave in F must vanish, and that the pressure at E, which is necessarily the same for all the tubes, must be constant. The waves in B and C are thus reflected as from an open end. That the conditions of the question are thus satisfied may also be seen by supposing a barrier taken across the tube F in the neighbourhood of E in such a way that the tubes B and C communicate without a change of section. The wave in each tube will then pass on into the other without interruption, and the pressure-variation at E, being the resultant of equal and opposite components, will vanish. This being so, the barrier may be removed without altering the conditions, and no wave will be propagated along F, whatever its section may be. The arrange-

Poisson, *Mém. d. l'Institut*, t. II. p. 305. The reader will not forget that both diameters must be small in comparison with the wave-length.

ment now under consideration was invented by Herschel, and has been employed by Quincke and others for experimental purposes,—an application that we shall afterwards have occasion to describe. The phenomenon itself is often referred to as an example of interference, to which there can be no objection, but the same cannot be said when the reader is led to suppose that the positive waves neutralise each other in F, and that there the matter ends. It must never be forgotten that there is no loss of energy in interference, but only a different distribution; when energy is diverted from one place, it reappears in another. In the present case the positive wave in A conveys energy with it. If there is no wave along F, there are two possible alternatives. Either energy accumulates in the branches, or else it passes back along A in the form of a negative wave. In order to see what really happens, let us trace the progress of the waves reflected back at E.

These waves are equal in magnitude and start from E in opposite phases; in the passage from E to D one has to travel a greater distance than the other by an odd multiple of $\frac{1}{2}\lambda$; and therefore on arrival at D they will be in complete accordance. Under these circumstances they combine into a single wave, which travels negatively along A, and there is no reflection. When the negative wave reaches the end of the tube A, or is otherwise disturbed in its course, the whole or a part may be reflected, and then the process is repeated. But however often this may happen there will be no wave along F, unless by accumulation in consequence of a coincidence of periods, the vibration in the branches become so great that a small fraction of it can no longer be neglected.

Fig. 56.

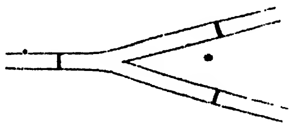


Or we may reason thus. Suppose the tube F cut off by a

barrier as before. The motion in the ring being due to forces acting at D is necessarily symmetrical with respect to D, and D'—the point which divides $DBCD$ into equal parts. Hence D' is a node, and the vibration is stationary. This being the case, at a point E distant $\frac{1}{4}\lambda$ from D' on either side, there must be a loop; and if the barrier be removed there will still be no tendency to produce vibration in F. If the perimeter of the ring be a multiple of λ, there may be vibration within it of the period in question, independently of any lateral openings.

Any combination of connected tubes may be treated in a similar manner. The general principle is that at any junction a space can be taken large enough to include all the region through which

Fig. 57.



the want of uniformity affects the law of the waves, and yet so small that its longest dimension may be neglected in comparison with λ. Under these circumstances the fluid within the space in question may be treated as if the wave-length were infinite, or the fluid itself incompressible, in which case its velocity-potential would satisfy $\nabla^2\phi = 0$, following the same laws as electricity.

265. When the section of a pipe is variable, the problem of the vibrations of air within it cannot generally be solved. The case of conical pipes will be treated on a future page. At present we will investigate an approximate expression for the pitch of a nearly cylindrical pipe, taking first the case where both ends are closed. The method that will be employed is similar to that used for a string whose density is not quite constant, §§ 91, 140, depending on the principle that the period of a free vibration fulfils the stationary condition, and may therefore be calculated from the potential and kinetic energies of any hypothetical motion not departing far from the actual type. In accordance with this plan we shall assume that the velocity normal to any section S is constant over the section, as must be very nearly the case when the variation of S is slow. Let X represent the total transfer of fluid at time t across the

section at x, reckoned from the equilibrium condition; then $\dot{X}$ represents the total velocity of the current, and $\dot{X} \div S$ represents the actual velocity of the particles of fluid, so that the kinetic energy of the motion within the tube is expressed by

$$T = \tfrac{1}{2}\rho \int \frac{\dot{X}^2}{S}\,dx \quad\text{.........................(1).}$$

The potential energy § 245 (12) is expressed in general by

$$V = \tfrac{1}{2}a^2\rho \iiint s^2\,dV,$$

or, since $dV = S\,dx$, by

$$V = \tfrac{1}{2}a^2\rho \int S s^2\,dx \quad\text{.........................(2).}$$

Again, by the condition of continuity,

$$-s = \frac{1}{S}\frac{dX}{dx} \quad\text{.........................(3),}$$

and thus

$$V = \tfrac{1}{2}a^2\rho \int \frac{1}{S}\left(\frac{dX}{dx}\right)^2 dx \quad\text{.........................(4).}$$

If we now assume for X an expression of the same form as would obtain, if S were constant, viz.

$$X = \sin\frac{\pi x}{l}\cos nt \quad\text{.........................(5),}$$

we obtain from the values of T and V in (1) and (4),

$$n^2 = \frac{a^2\pi^2}{l^2}\int_0^l \cos^2\frac{\pi x}{l}\frac{dx}{S} \div \int_0^l \sin^2\frac{\pi x}{l}\frac{dx}{S} \quad\text{.................(6),}$$

or, if we write $S = S_0 + \Delta S$ and neglect the square of ΔS,

$$n^2 = \frac{a^2\pi^2}{l^2}\left\{1 - 2\int_0^l \cos\frac{2\pi x}{l}\frac{\Delta S}{S_0}\frac{dx}{l}\right\} \quad\text{.................(7).}$$

The result may be expressed conveniently in terms of Δl, the correction that must be made to l in order that the pitch may be calculated from the ordinary formula, as if S were constant. For the value of Δl we have

$$\Delta l = \int_0^l \cos\frac{2\pi x}{l}\frac{\Delta S}{S_0}\,dx \quad\text{.................(8).}$$

The effect of a variation of section is greatest near a node or near a loop. An enlargement of section in the first case lowers the pitch, and in the second case raises it. At the points midway between the nodes and loops a slight variation of section is without effect. The pitch is thus decidedly altered by an enlargement or contraction near the middle of the tube, but the influence of a slight conicality would be much less.

The expression for Δl in (8) is applicable as it stands to the gravest tone only; but we may apply it to the m^{th} tone of the harmonic scale, if we modify it by the substitution of $\cos \dfrac{2m\pi x}{l}$ for $\cos \dfrac{2\pi x}{l}$.

In the case of a tube *open* at both ends (5) is replaced by

$$X = \cos \frac{\pi x}{l} \cos nt \qquad\qquad (9),$$

which leads to

$$\Delta l = -\int_0^l \cos \frac{2\pi x}{l} \frac{\Delta S}{S_0} dx \qquad\qquad (10),$$

instead of (8). The pitch of the sound is now raised by an enlargement at the ends, or by a contraction at the middle, of the tube; and, as before, it is unaffected by a slight general conicality (§ 281).

266. The case of progressive waves moving in a tube of variable section is also interesting. In its general form the problem would be one of great difficulty; but where the change of section is very gradual, so that no considerable alteration occurs within a distance of a great many wave-lengths, the principle of energy will guide us to an approximate solution. It is not difficult to see that in the case supposed there will be no sensible reflection of the wave at any part of its course, and that therefore the energy of the motion must remain unchanged[1]. Now we know, § 245, that for a given area of wave-front, the energy of a train of simple waves is as the square of the amplitude, from which it follows that as the waves advance the amplitude of vibration varies inversely as the square root of the section of the tube. In all other respects the type of vibration remains absolutely unchanged. From these results we may get a general idea of the action of an ear-trumpet.

[1] *Phil. Mag.* (5) I. p. 261.

It appears that according to the ordinary approximate equations, there is no limit to the concentration of sound producible in a tube of gradually diminishing section.

The same method is applicable, when the density of the medium varies slowly from point to point. For example, the amplitude of a sound-wave moving upwards in the atmosphere may be determined by the condition that the energy remains unchanged. From § 245 it appears that the amplitude is inversely as the square root of the density[1].

[1] A delicate question arises as to the ultimate fate of sonorous waves propagated upwards. It should be remarked that in rare air the deadening influence of viscosity is much increased.

CHAPTER XIII.

SPECIAL PROBLEMS. REFLECTION AND REFRACTION OF PLANE WAVES.

267. Before undertaking the discussion of the general equations for aërial vibrations we may conveniently turn our attention to a few special problems, relating principally to motion in two dimensions, which are susceptible of rigorous and yet comparatively simple solution. In this way the reader, to whom the subject is new, will acquire some familiarity with the ideas and methods employed before attacking more formidable difficulties.

In the previous chapter (§ 255) we investigated the vibrations in one dimension, which may take place parallel to the axis of a tube, of which both ends are closed. We will now inquire what vibrations are possible within a closed rectangular box, dispensing with the restriction that the motion is to be in one dimension only. For each simple vibration, of which the system is capable, ϕ varies as a circular function of the time, say $\cos \kappa at$, where κ is some constant; hence $\ddot{\phi} = -\kappa^2 a^2 \phi$, and therefore by the general differential equation (9) § 244

$$\nabla^2\phi + \kappa^2\phi = 0 \quad\text{.........................}\quad (1).$$

Equation (1) must be satisfied throughout the whole of the included volume. The surface condition to be satisfied over the six sides of the box is simply

$$\frac{d\phi}{dn} = 0 \quad\text{.............................}\quad (2),$$

where dn represents an element of the normal to the surface. It is only for special values of κ that it is possible to satisfy (1) and (2) simultaneously.

Taking three edges which meet as axes of rectangular co-ordinates, and supposing that the lengths of the edges are respectively α, β, γ, we know (§ 255) that

$$\phi = \cos\left(p\frac{\pi x}{\alpha}\right), \quad \phi = \cos\left(q\frac{\pi y}{\beta}\right), \quad \phi = \cos\left(r\frac{\pi z}{\gamma}\right),$$

where p, q, r are integers, are particular solutions of the problem. By any of these forms equation (2) is satisfied, and provided that κ be equal to $p\frac{\pi}{\alpha}$, $q\frac{\pi}{\beta}$, or $r\frac{\pi}{\gamma}$, as the case may be, (1) is also satisfied. It is equally evident that the boundary equation (2) is satisfied over all the surface by the form

$$\phi = \cos\left(p\frac{\pi x}{\alpha}\right)\cos\left(q\frac{\pi y}{\beta}\right)\cos\left(r\frac{\pi z}{\gamma}\right) \quad\text{............ (3),}$$

a form which also satisfies (1), if κ be taken such that

$$\kappa^2 = \pi^2\left(\frac{p^2}{\alpha^2} + \frac{q^2}{\beta^2} + \frac{r^2}{\gamma^2}\right) \quad\text{...................... (4),}$$

where as before p, q, r are integers.

The general solution, obtained by compounding all particular solutions included under (3), is

$$\phi = \Sigma\Sigma\Sigma\,(A\cos\kappa at + B\sin\kappa at)$$
$$\times\cos\left(p\frac{\pi x}{\alpha}\right)\cos\left(q\frac{\pi y}{\beta}\right)\cos\left(r\frac{\pi z}{\gamma}\right) \quad\text{..............(5),}$$

in which A and B are arbitrary constants, and the summation is extended to all integral values of p, q, r.

This solution is sufficiently general to cover the case of any initial state of things within the box, not involving molecular rotation. The initial distribution of velocities depends upon the initial value of ϕ, or $\int(u_0dx + v_0dy + w_0dz)$, and by Fourier's theorem can be represented by (5), suitable values being ascribed to the coefficients A. In like manner an arbitrary initial distribution of condensation (or rarefaction), depending on the initial value of $\dot\phi$, can be represented by ascribing suitable values to the coefficients B.

The investigation might be presented somewhat differently by commencing with assuming in accordance with Fourier's

theorem that the general value of ϕ at time t can be expressed in the form

$$\phi = \Sigma\,\Sigma\,\Sigma\, C \cos\left(p\frac{\pi x}{\alpha}\right)\cos\left(q\frac{\pi y}{\beta}\right)\cos\left(r\frac{\pi z}{\gamma}\right),$$

in which the coefficients C may depend upon t, but not upon x, y, z. The expressions for T and V would then be formed, and shewn to involve only the squares of the coefficients C, and from these expressions would follow the normal equations of motion connecting each normal co-ordinate C with the time.

The gravest mode of vibration is that in which the entire motion is parallel to the longest dimension of the box, and there is no internal node. Thus, if α be the greatest of the three sides α, β, γ, we are to take $p=1$, $q=0$, $r=0$.

In the case of a cubical box, $\alpha=\beta=\gamma$, and then instead of (4) we have

$$\kappa^2 = \frac{\pi^2}{\alpha^2}(p^2+q^2+r^2) \quad\text{.....................}\quad (6),$$

or, if λ be the wave-length of plane waves of the same period,

$$\lambda = 2\alpha \div \sqrt{(p^2+q^2+r^2)} \quad\text{.....................}\quad (7).$$

For the gravest mode $p=1$, $q=0$, $r=0$, or $p=0$, $q=1$, $r=0$, &c., and $\lambda=2\alpha$. The next gravest is when $p=1$, $q=1$, $r=0$, &c., and then $\lambda=\sqrt{2}\,\alpha$. When $p=1$, $q=1$, $r=1$, $\lambda=\dfrac{2}{\sqrt{3}}\alpha$. For the fourth gravest mode $p=2$, $q=0$, $r=0$, &c., and then $\lambda=4\alpha$.

As in the case of the membrane (§ 197), when two or more primitive modes have the same period of vibration, other modes of like period may be derived by composition.

The trebly infinite series of possible simple component vibrations is not necessarily completely represented in particular cases of compound vibrations. If, for example, we suppose the contents of the box in its initial condition to be neither condensed nor rarefied in any part, and to have a uniform velocity, whose components parallel to the axes of co-ordinates are respectively u_0, v_0, w_0, no simple vibrations are generated for which more than one of the three numbers p, q, r is finite. In fact each component initial velocity may be considered separately, and the problem is similar to that solved in § 258.

In future chapters we shall meet with other examples of the vibrations of air within completely closed vessels.

Some of the natural notes of the air contained within a room may generally be detected on singing the scale. Probably it is somewhat in this way that blind people are able to estimate the size of rooms[1].

In long and narrow passages the vibrations parallel to the length are too slow to affect the ear, but notes due to transverse vibrations may often be heard. The relative proportions of the various overtones depend upon the place at which the disturbance is created[2].

In some cases of this kind the pitch of the vibrations, whose direction is principally transverse, is influenced by the occurrence of longitudinal motion. Suppose, for example, in (3) and (4), that $q=1$, $r=0$, and that α is much greater than β. For the principal transverse vibration $p=0$, and $\kappa=\pi\div\beta$. But besides this there are other modes of vibration in which the motion is principally transverse, obtained by ascribing to p small integral values. Thus, when $p=1$,

$$\kappa^2=\pi^2\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right),$$

shewing that the pitch is nearly the same as before[3].

268. If we suppose γ to become infinitely great, the box of the preceding section is transformed into an infinite rectangular tube, whose sides are α and β. Whatever may be the motion of the air within this tube, its velocity-potential may be expressed by Fourier's theorem in the series

$$\phi=\Sigma\,\Sigma\,A_{pq}\cos\frac{p\pi x}{\alpha}\cos\frac{q\pi y}{\beta}\;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

[1] A remarkable instance is quoted in Young's *Natural Philosophy*, II. p. 272, from Darwin's *Zoonomia*, II. 487. "The late blind Justice Fielding walked for the first time into my room, when he once visited me, and after speaking a few words said, 'This room is about 22 feet long, 18 wide, and 12 high'; all which he guessed by the ear with great accuracy."

[2] Oppel, *Die harmonischen Obertöne des durch parallele Wände erregten Reflexionstones. Fortschritte der Physik*, XX. p. 130.

[3] There is an underground passage in my house in which it is possible, by singing the right note, to excite free vibrations of many seconds' duration, and it often happens that the resonant note is affected with distinct beats. The breadth of the passage is about 4 feet, and the height about 6½ feet.

where the coefficients A are independent of x and y. By the use of this form we secure the fulfilment of the boundary condition that there is to be no velocity across the sides of the tube; the nature of A as a function of z and t depends upon the other conditions of the problem.

Let us consider the case in which the motion at every point is harmonic, and due to a normal motion imposed upon a barrier stretching across the tube at $z=0$. Assuming ϕ to be proportional to $e^{i\kappa at}$ at all points, we have the usual differential equation

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}+\kappa^2\phi=0 \quad\text{.................}(2),$$

which by the conjugate property of the functions must be satisfied separately by each term of (1). Thus to determine A_{pq} as a function of z, we get

$$\frac{d^2A_{pq}}{dz^2}+\left[\kappa^2-\pi^2\left(\frac{p^2}{\alpha^2}+\frac{q^2}{\beta^2}\right)\right]A_{pq}=0 \quad\text{...........}(3).$$

The solution of this equation differs in form according to the sign of the coefficient of A_{pq}. When p and q are both zero, the coefficient is necessarily positive, but as p and q increase the coefficient changes sign. If the coefficient be positive and be called μ^2, the general value of A_{pq} may be written

$$A_{pq}=B_{pq}\,e^{i(\kappa at+\mu z)}+C_{pq}\,e^{i(\kappa at-\mu z)} \quad\text{..............}(4),$$

where, as the factor $e^{i\kappa at}$ is expressed, B_{pq}, C_{pq} are absolute constants. However, the first term in (4) expresses a motion propagated in the negative direction, which is excluded by the conditions of the problem, and thus we are to take simply as the term corresponding to p, q,

$$\phi=C_{pq}\cos\frac{p\pi x}{\alpha}\cos\frac{q\pi y}{\beta}\,e^{i(\kappa at-\mu z)}.$$

In this expression C_{pq} may be complex; passing to real quantities and taking two new real arbitrary constants, we obtain

$$\phi=[D_{pq}\cos(\kappa at-\mu z)+E_{pq}\sin(\kappa at-\mu z)]\cos\frac{p\pi x}{\alpha}\cos\frac{q\pi y}{\beta} \quad\text{...}(5).$$

We have now to consider the form of the solution in cases where the coefficient of A_{pq} in (3) is negative. If we call it $-\nu^2$, the solution corresponding to (4) is

$$A_{pq}=e^{i\kappa at}\left(B_{pq}\,e^{\nu z}+C_{pq}\,e^{-\nu z}\right) \quad\text{.................}(6).$$

of which the first term is to be rejected as becoming infinite with z. We thus obtain corresponding to (5)

$$\phi = e^{-\nu z}[D_{pq}\cos\kappa at + E_{pq}\sin\kappa at]\cos\frac{p\pi x}{\alpha}\cos\frac{q\pi y}{\beta}\ \ldots\ldots(7).$$

The solution obtained by combining all the particular solutions given by (5) and (7) is the general solution of the problem, and allows of a value of $\frac{d\phi}{dz}$ over the section $z=0$, arbitrary at every point in both amplitude and phase.

At a great distance from the source the terms given in (7) become insensible, and the motion is represented by the terms of (5) alone. The effect of the terms involving high values of p and q is thus confined to the neighbourhood of the source, and at moderate distances any sudden variations or discontinuities in the motion at $z=0$ are gradually eased off and obliterated.

If we fix our attention on any particular simple mode of vibration (for which p and q do not both vanish), and conceive the frequency of vibration to increase from zero upwards, we see that the effect, at first confined to the neighbourhood of the source, gradually extends further and further, and after a certain value is passed, propagates itself to an infinite distance, the critical frequency being that of the two dimensional free vibrations of the corresponding mode. Below the critical point no work is required to *maintain* the motion; above it as much work must be done at $z=0$ as is carried off to infinity in the same time.

269. We will now examine the result of the composition of two trains of plane waves of harmonic type, whose amplitudes and wave-lengths are equal, but whose directions of propagation are inclined to one another at an angle 2α. The problem is one of two dimensions only, inasmuch as everything is the same in planes perpendicular to the lines of intersection of the two sets of wave-fronts.

At any moment of time the positions of the planes of maximum condensation for each train of waves may be represented by parallel lines drawn at equal intervals λ on the plane of the paper, and these lines must be supposed to move with a velocity a in a direction perpendicular to their length. If both sets of lines be drawn, the paper will be divided into a system of equal parallelo-

grams, which advance in the direction of one set of diagonals. At each corner of a parallelogram the condensation is doubled by the superposition of the two trains of waves, and in the centre of each parallelogram the rarefaction is a maximum for the same reason. On each diagonal there is therefore a series of maxima and minima condensations, advancing without change of relative position and with velocity $a \div \cos\alpha$. Between each adjacent pair of lines of maxima and minima there is a parallel line of zero condensation, on which the two trains of waves neutralize one another. It is especially remarkable that, if the wave-pattern were visible (like the corresponding water wave-pattern to which the whole of the preceding argument is applicable), it would appear to move forwards without change of type in a direction different from that of either component train, and with a velocity different from that with which both component trains move.

In order to express the result analytically, let us suppose that the two directions of propagation are equally inclined at an angle α to the axis of x. The condensations themselves may be denoted by

$$\cos\frac{2\pi}{\lambda}(at - x\cos\alpha - y\sin\alpha)$$

and

$$\cos\frac{2\pi}{\lambda}(at - x\cos\alpha + y\sin\alpha)$$

respectively, and thus the expression for the resultant is

$$s = \cos\frac{2\pi}{\lambda}(at - x\cos\alpha - y\sin\alpha) + \cos\frac{2\pi}{\lambda}(at - x\cos\alpha + y\sin\alpha)$$

$$= 2\cos\frac{2\pi}{\lambda}(at - x\cos\alpha)\cos\frac{2\pi}{\lambda}(y\sin\alpha)\text{.........(1).}$$

It appears from (1) that the distribution of s on the plane xy advances parallel to the axis of x, unchanged in type, and with a uniform velocity $a \div \cos\alpha$. Considered as depending on y, s is a maximum, when $y\sin\alpha$ is equal to $0, \lambda, 2\lambda, 3\lambda$, &c., while for the intermediate values, viz. $\frac{1}{2}\lambda, \frac{3}{2}\lambda$, &c., s vanishes.

If $\alpha = \frac{1}{2}\pi$, so that the two trains of waves meet one another directly, the velocity of propagation parallel to x becomes infinite, and (1) assumes the form

$$s = 2\cos\left(\frac{2\pi}{\lambda}at\right)\cos\left(\frac{2\pi}{\lambda}y\right)\text{..................(2);}$$

which represents *stationary* waves.

The problem that we have just been considering is in reality the same as that of the reflection of a train of plane waves by an infinite plane wall. Since the expression on the right-hand side of equation (1) is an even function of y, s is symmetrical with respect to the axis of x, and consequently there is no motion across that axis. Under these circumstances it is evident that the motion could in no way be altered by the introduction along the axis of x of an absolutely immovable wall. If α be the angle between the surface and the direction of propagation of the incident waves, the velocity with which the places of maximum condensation (corresponding to the greatest elevation of water-waves) move along the wall is $a \div \cos \alpha$. It may be noticed that the aërial pressures have no tendency to move the wall as a whole, except in the case of absolutely perpendicular incidence, since they are at any moment as much negative as positive.

270. So long as the medium which is the vehicle of sound continues of unbroken uniformity, plane waves may be propagated in any direction with constant velocity and with type unchanged; but a disturbance ensues when the waves reach any part where the mechanical properties of the medium undergo a change. The general problem of the vibrations of a variable medium is probably quite beyond the grasp of our present mathematics, but many of the points of physical interest are raised in the case of plane waves. Let us suppose that the medium is uniform above and below a certain infinite plane ($x = 0$), but that in crossing that plane there is an abrupt variation in the mechanical properties on which the propagation of sound depends—namely the *compressibility* and the *density*. On the upper side of the plane (which for distinctness of conception we may suppose horizontal) a train of plane waves advances so as to meet it more or less obliquely; the problem is to determine the (refracted) wave which is propagated onwards within the second medium, and also that thrown back into the first medium, or reflected. We have in the first place to form the equations of motion and to express the boundary conditions.

In the upper medium, if ρ be the natural density and s the condensation,

$$\text{density} = \rho\,(1 + s),$$

and

$$\text{pressure} = P\,(1 + As),$$

where A is a coefficient depending on the compressibility, and P is the undisturbed pressure. In like manner in the lower medium

$$\text{density} = \rho_1 (1 + s_1),$$

$$\text{pressure} = P(1 + A_1 s_1),$$

the undisturbed pressure being the same on both sides of $x = 0$. Taking the axis of z parallel to the line of intersection of the plane of the waves with the surface of separation $x = 0$, we have for the upper medium (§ 244),

$$\frac{d^2\phi}{dt^2} = V^2 \left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right) \quad \text{...................(1)},$$

and
$$\frac{d\phi}{dt} + V^2 s = 0 \quad \text{........................(2)},$$

where
$$V^2 = PA \div \rho \quad \text{..........................(3)}.$$

Similarly, in the lower medium,

$$\frac{d^2\phi_1}{dt^2} = V_1^2 \left(\frac{d^2\phi_1}{dx^2} + \frac{d^2\phi_1}{dy^2}\right) \quad \text{.....................(4)},$$

and
$$\frac{d\phi_1}{dt} + V_1^2 s_1 = 0 \quad \text{......................(5)},$$

where
$$V_1^2 = PA_1 \div \rho_1 \quad \text{........................(6)}.$$

These equations must be satisfied at all points of the fluid. Further the boundary conditions require (α) that at all points of the surface of separation the velocities perpendicular to the surface must be the same for the two fluids, or

$$\frac{d\phi}{dx} = \frac{d\phi_1}{dx}, \text{ when } x = 0 \quad \text{................(7)};$$

(β) that the pressures must be the same, whence $A_1 s_1 = As$, or by (2), (3), (5) and (6),

$$\rho \frac{d\phi}{dt} = \rho_1 \frac{d\phi_1}{dt}, \text{ when } x = 0 \quad \text{.............(8)}.$$

In order to represent a train of waves of harmonic type, we may assume ϕ and ϕ_1 to be proportional to $e^{i(ax+by+ct)}$, where $ax + by = \text{const.}$ gives the direction of the plane of the waves. If we assume for the incident wave,

$$\phi = \phi' \, e^{i(ax+by+ct)} \quad \text{........................(9)},$$

the reflected and refracted waves may be represented respectively by

$$\phi=\phi'' e^{i(-ax+by+ct)} \qquad\qquad (10),$$

$$\phi_1=\phi_1 e^{i(a_1x+by+ct)} \qquad\qquad (11).$$

The coefficient of t is necessarily the same in all three waves on account of the periodicity, and the coefficient of y must be the same, since the traces of all the waves on the plane of separation must move together. With regard to the coefficient of x, it appears by substitution in the differential equations that its sign is changed in passing from the incident to the reflected wave; in fact

$$c^2=V^2[(\pm a)^2+b^2]=V_1^2[a_1^2+b^2] \qquad\qquad (12).$$

Now $b\div\sqrt{(a^2+b^2)}$ is the sine of the angle included between the axis of x and the normal to the plane of the waves—in optical language, the sine of the angle of incidence, and $b\div\sqrt{(a_1^2+b^2)}$ is in like manner the sine of the angle of refraction. If these angles be called θ, θ_1, (12) asserts that $\sin\theta : \sin\theta_1$ is equal to the constant ratio $V:V_1$,—the well-known law of sines. The laws of refraction and reflection follow simply from the fact that the velocity of propagation normal to the wave-fronts is constant in each medium, that is to say, independent of the *direction* of the wave-front, taken in connection with the equal velocities of the traces of all the waves on the plane of separation ($V\div\sin\theta=V_1\div\sin\theta_1$). It remains to satisfy the boundary conditions (7) and (8).

These give

$$\left.\begin{array}{l} a(\phi'-\phi'')=a_1\phi_1 \\ \rho(\phi'+\phi'')=\rho_1\phi_1 \end{array}\right\} \qquad\qquad (13),$$

whence

$$2\phi'=\left(\frac{\rho_1}{\rho}+\frac{a_1}{a}\right)\phi_1; \qquad 2\phi''=\left(\frac{\rho_1}{\rho}-\frac{a_1}{a}\right)\phi_1 \qquad\qquad (14).$$

This completes the symbolical solution. If a_1 (and θ_1) be real, we see that if the incident wave be

$$\phi=\cos(ax+by+ct),$$

or in terms of V, λ, and θ,

$$\phi=\cos\frac{2\pi}{\lambda}(x\cos\theta+y\sin\theta+Vt) \qquad\qquad (15),$$

the reflected wave is

$$\phi = \frac{\dfrac{\rho_1}{\rho} - \dfrac{\cot\theta_1}{\cot\theta}}{\dfrac{\rho_1}{\rho} + \dfrac{\cot\theta_1}{\cot\theta}} \cos\frac{2\pi}{\lambda}(-x\cos\theta + y\sin\theta + Vt) \;\ldots(16),$$

and the refracted wave is

$$\phi_1 = \frac{2}{\dfrac{\rho_1}{\rho} + \dfrac{\cot\theta_1}{\cot\theta}} \cos\frac{2\pi}{\lambda_1}(x\cos\theta_1 + y\sin\theta_1 + V_1 t) \;\ldots(17).$$

The formula for the amplitude of the reflected wave, viz.

$$\frac{\phi''}{\phi'} = \frac{\dfrac{\rho_1}{\rho} - \dfrac{\cot\theta_1}{\cot\theta}}{\dfrac{\rho_1}{\rho} + \dfrac{\cot\theta_1}{\cot\theta}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

is here obtained on the supposition that the waves are of harmonic type; but since it does not involve λ, and there is no change of phase, it may be extended by Fourier's theorem to waves of any type whatever.

If there be no reflected wave, $\cot\theta_1 : \cot\theta = \rho_1 : \rho$, from which and $(1 + \cot^2\theta_1) : (1 + \cot^2\theta) = V^2 : V_1^2$, we deduce

$$\left(\frac{\rho_1^2}{\rho^2} - \frac{V^2}{V_1^2}\right)\cot^2\theta = \frac{V^2}{V_1^2} - 1 \;\ldots\ldots\ldots\ldots\ldots(19),$$

which shews that, provided the refractive index $V_1 : V$ be intermediate in value between unity and $\rho : \rho_1$, there is always an angle of incidence at which the wave is completely intromitted; but otherwise there is no such angle.

Since (18) is not altered (except as to sign) by an interchange of θ, θ_1; ρ, ρ_1; &c., we infer that a wave incident in the second medium at an angle θ_1 is reflected in the same proportion as a wave incident in the first medium at an angle θ.

As a numerical example let us suppose that the upper medium is air at atmospheric pressure, and the lower medium water. Substituting for $\cot\theta_1$ its value in terms of θ and the refractive index, we get

$$\frac{\cot\theta_1}{\cot\theta} = \frac{V}{V_1}\sqrt{1 - \left(\frac{V_1^2}{V^2} - 1\right)\tan^2\theta} \;\ldots\ldots\ldots\ldots(20),$$

or, since $V_1 : V = 4 \cdot 3$ approximately,

$$\frac{\cot \theta_1}{\cot \theta} = \cdot 23 \sqrt{(1 - 17 \cdot 5 \tan^2 \theta)},$$

which shews that the ratio of cotangents diminishes to zero, as θ increases from zero to about 13°, after which it becomes imaginary, indicating total reflection, as we shall see presently. It must be remembered that in applying optical terms to acoustics, it is the *water* that must be conceived to be the 'rare' medium. The ratio of densities is about 770 : 1; so that

$$\frac{\phi''}{\phi'} = \frac{1 - \cdot 0003 \sqrt{1 - 17 \cdot 5 \tan^2 \theta}}{1 + \cdot 0003 \sqrt{1 - 17 \cdot 5 \tan^2 \theta}}$$
$$= 1 - \cdot 0006 \sqrt{1 - 17 \cdot 5 \tan^2 \theta} \text{ very nearly.}$$

Even at perpendicular incidence the reflection is sensibly perfect.

If both media be gaseous, $A_1 = A$, if the temperature be constant; and even if the development of heat by compression be taken into account, there will be no sensible difference between A and A_1 in the case of the simple gases. Now, if $A_1 = A$, $\rho_1 : \rho = \sin^2 \theta : \sin^2 \theta_1$, and the formula for the intensity of the reflected wave becomes

$$\frac{\phi''}{\phi'} = \frac{\sin 2\theta - \sin 2\theta_1}{\sin 2\theta + \sin 2\theta_1} = \frac{\tan(\theta - \theta_1)}{\tan(\theta + \theta_1)} \quad \text{..........} \quad (21),$$

coinciding with that given by Fresnel for light polarized perpendicularly to the plane of incidence. In accordance with Brewster's law the reflection vanishes at the angle of incidence, whose tangent is $V \div V_1$.

But, if on the other hand $\rho_1 = \rho$, the cause of disturbance being the change of compressibility, we have

$$\frac{\phi''}{\phi'} = \frac{\tan \theta_1 - \tan \theta}{\tan \theta_1 + \tan \theta} = \frac{\sin(\theta_1 - \theta)}{\sin(\theta_1 + \theta)} \quad \text{.............} \quad (22),$$

agreeing with Fresnel's formula for light polarized in the plane of incidence. In this case the reflected wave does not vanish at any angle of incidence.

In general, when $\theta = 0$,

$$\phi'' : \phi' = \frac{\rho_1}{\rho} - \frac{V}{V_1} : \frac{\rho_1}{\rho} + \frac{V}{V_1} \quad \text{..................} \quad (23);$$

so that there is no reflection, if $\rho_1 : \rho = V : V_1$. In the case of gases $V^2 : V_1^2 = \rho_1 : \rho$, and then

$$\frac{\phi''}{\phi'} = \frac{\sqrt{\rho_1} - \sqrt{\rho}}{\sqrt{\rho_1} + \sqrt{\rho}} = \frac{V - V_1}{V + V_1} \quad \text{.................. (24).}$$

Suppose, for example, that after perpendicular incidence reflection takes place at a surface separating air and hydrogen. We have

$$\rho = \cdot001276, \quad \rho_1 = \cdot00008837;$$

whence $\sqrt{\rho} : \sqrt{\rho_1} = 3\cdot800$, giving

$$\phi'' = -\cdot5833\,\phi'.$$

The ratio of intensities, which is as the square of the amplitudes, is $\cdot3402 : 1$, so that about one-third part is reflected.

If the difference between the two media be very small, and we write $V_1 = V + \delta V$, (24) becomes

$$\frac{\phi''}{\phi'} = -\tfrac{1}{2}\frac{\delta V}{V} \quad \text{........................ (25).}$$

If the first medium be air at 0° Cent., and the second medium be air at t° Cent., $V + \delta V = V\sqrt{1 + \cdot00366\,t}$; so that

$$\frac{\phi''}{\phi'} = -\cdot00091t.$$

The ratio of the intensities of the reflected and incident sounds is therefore $\cdot83 \times 10^{-6} \times t^2 : 1$.

As another example of the same kind we may take the case in which the first medium is dry air and the second is air of the same temperature saturated with moisture. At 10° Cent. air saturated with moisture is lighter than dry air by about one part in 220, so that $\delta V = \frac{V}{440}$ nearly. Hence we conclude from (25) that the reflected sound is only about one 774,000th part of the incident sound.

From these calculations we see that reflections from warm or moist air must generally be very small, though of course the effect may accumulate by repetition. It must also be remembered that in practice the transition from one state of things to the other would be gradual, and not abrupt, as the present theory supposes. If the space occupied by the transition amount to a considerable

fraction of the wave-length, the reflection would be materially lessened. On this account we might expect grave sounds to travel through a heterogeneous medium less freely than acute sounds.

The reflection of sound from surfaces separating portions of gas of different densities has engaged the attention of Prof. Tyndall, who has devised several striking experiments in illustration of the subject[1]. For example, sound from a high-pitched reed was conducted through a tin tube towards a sensitive flame, which served as an indicator. By the interposition of a coal-gas flame issuing from an ordinary bat's-wing burner between the tube and the sensitive flame, the greater part of the effect could be cut off. Not only so, but by holding the flame at a suitable angle, the sound could be reflected through another tube in sufficient quantity to excite a second sensitive flame, which but for the interposition of the reflecting flame would have remained undisturbed.

The preceding expressions (16), (17), (18) hold good in every case of reflection from a 'denser' medium; but if the velocity of sound be greater in the lower medium, and the angle of incidence exceed the critical angle, a_1 becomes imaginary, and the formulæ require modification. In the latter case it is impossible that a refracted wave should exist, since, even if the angle of refraction were 90°, its trace on the plane of separation must necessarily outrun the trace of the incident wave.

If $-ia_1'$ be written in place of a_1, the symbolical equations are

Incident wave

$$\phi = e^{i(ax+by+ct)},$$

Reflected wave

$$\phi = \frac{\dfrac{\rho_1}{\rho} + i\dfrac{a_1'}{a}}{\dfrac{\rho_1}{\rho} - i\dfrac{a_1'}{a}}\, e^{i(-ax+by+ct)},$$

Refracted wave

$$\phi_1 = \frac{2}{\dfrac{\rho_1}{\rho} - i\dfrac{a_1'}{a}}\, e^{i(-ia_1'x+by+ct)};$$

from which by discarding the imaginary parts, we obtain

[1] *Sound*, 3rd edition, p. 282.

Incident wave

$$\phi = \cos(ax + by + ct) \quad\text{.....................(26)},$$

Reflected wave

$$\phi = \cos(-ax + by + ct + 2\epsilon) \quad\text{.............(27)},$$

Refracted wave

$$\phi = \frac{2}{\left(\frac{\rho_1^2}{\rho^2} + \frac{a_1'^2}{a^2}\right)^{\frac{1}{2}}} e^{a_1'x} \cos(by + ct + \epsilon) \quad\text{..........(28)},$$

where
$$\tan\epsilon = \frac{a_1'\rho}{a\rho_1} \quad\text{.........................(29)}.$$

These formulæ indicate total reflection. The disturbance in the second medium is not a wave at all in the ordinary sense, and at a short distance from the surface of separation (x negative) becomes insensible. Calculating a_1' from (12) and expressing it in terms of θ and λ, we find

$$a_1' = \frac{2\pi}{\lambda}\sqrt{\sin^2\theta - \frac{V^2}{V_1^2}} \quad\text{.................(30)},$$

shewing that the disturbance does not penetrate into the second medium more than a few wave-lengths.

The difference of phase between the reflected and the incident waves is 2ϵ, where

$$\tan\epsilon = \frac{\rho}{\rho_1}\sqrt{\tan^2\theta - \frac{V^2}{V_1^2}\sec^2\theta} \quad\text{............(31)}.$$

If the media have the same compressibilities, $\rho : \rho_1 = V_1^2 : V^2$, and

$$\tan\epsilon = \frac{V_1}{V}\sqrt{\frac{V_1^2}{V^2}\tan^2\theta - \sec^2\theta} \quad\text{...........(32)}.$$

Since there is no loss of energy in reflection and refraction, the work transmitted in any time across any area of the front of the incident wave must be equal to the work transmitted in the same time across corresponding areas of the reflected and refracted waves. These corresponding areas are plainly in the ratio

$$\cos\theta : \cos\theta : \cos\theta_1;$$

and thus by § 245 (τ being the same for all the waves),

$$\cos\theta\,\frac{\rho}{V}(\phi'^2 - \phi''^2) = \cos\theta_1\,\frac{\rho_1}{V_1}\phi_1^2,$$

or since $$V : V_1 = \sin\theta : \sin\theta_1,$$

$$\rho\cot\theta\,(\phi'^2 - \phi''^2) = \rho_1\cot\theta_1\,\phi_1^2 \qquad (33),$$

which is the energy condition, and agrees with the result of multiplying together the two boundary equations (13).

When the velocity of propagation is greater in the lower than in the upper medium, and the angle of incidence exceeds the critical angle, no energy is transmitted into the second medium; in other words the reflection is total.

The method of the present investigation is substantially the same as that employed by Green in a paper on the Reflection and Refraction of Sound[1]. The case of perpendicular incidence was first investigated by Poisson[2], who obtained formulæ corresponding to (23) and (24), which had however been already given by Young for the reflection of Light. In a subsequent memoir[3] Poisson considered the general case of oblique incidence, limiting himself, however, to gaseous media for which Boyle's law holds good, and by a very complicated analysis arrived at a result equivalent to (21). He also verified that the energies of the reflected and refracted waves make up that of the incident wave.

271. If the second medium be indefinitely extended downwards with complete uniformity in its mechanical properties, the transmitted wave is propagated onwards continually. But if at $x = -l$ there be a further change in the compressibility, or density, or both, part of the wave will be thrown back, and on arrival at the first surface ($x = 0$) will be divided into two parts, one transmitted into the first medium, and one reflected back, to be again divided at $x = -l$, and so on. By following the progress of these waves the solution of the problem may be obtained, the resultant reflected and transmitted waves being compounded of an infinite convergent series of components, all parallel and harmonic. This is the method usually adopted in Optics for the corresponding problem, and is quite rigorous, though perhaps not always sufficiently explained; but it does not appear to have any advantage over a more straightforward analysis. In the following investigation we shall confine ourselves to the case where the third medium is similar in its properties to the first medium.

[1] *Cambridge Transactions*, 1838.

[2] *Mém. de l'Institut*, t. II. p. 305. 1819.

[3] "Mémoire sur le mouvement de deux fluides élastiques superposés." *Mém. de l'Institut*, t. X. p. 317. 1831.

In the first medium

$$\phi = \phi' e^{i(ax+by+ct)} + \phi'' e^{i(-ax+by+ct)}.$$

In the second medium

$$\psi = \psi' e^{i(a_1x+by+ct)} + \psi'' e^{i(-a_1x+by+ct)}.$$

In the third medium

$$\phi = \phi_1 e^{i(ax+by+ct)},$$

with the conditions

$$c^2 = V^2(a^2+b^2) = V_1^2(a_1^2+b^2) \quad \text{.................(1).}$$

At the two surfaces of separation we have to secure the equality of normal motions and pressures; for $x=0$,

$$\left.\begin{aligned} a(\phi'-\phi'') &= a_1(\psi'-\psi'') \\ \rho(\phi'+\phi'') &= \rho_1(\psi'+\psi'') \end{aligned}\right\} \quad \text{..................(2);}$$

for $x=-l$,

$$\left.\begin{aligned} a_1(\psi' e^{-ia_1l} - \psi'' e^{ia_1l}) &= a\phi_1 e^{-ial} \\ \rho_1(\psi' e^{-ia_1l} + \psi'' e^{ia_1l}) &= \rho\phi_1 e^{-ial} \end{aligned}\right\} \quad \text{...............(3),}$$

from which ψ' and ψ'' are to be eliminated. We get

$$\left.\begin{aligned} (\phi'-\phi'')\cos a_1l - i\frac{a_1\rho}{a\rho_1}(\phi'+\phi'')\sin a_1l &= \phi_1 e^{-ial} \\ (\phi'+\phi'')\cos a_1l - i\frac{a\rho_1}{a_1\rho}(\phi'-\phi'')\sin a_1l &= \phi_1 e^{-ial} \end{aligned}\right\} \quad \text{.........(4);}$$

and from these, if for brevity $\dfrac{a\rho_1}{a_1\rho} = \alpha$,

$$\frac{\phi''}{\phi'} = \frac{\alpha - \dfrac{1}{\alpha}}{\alpha + \dfrac{1}{\alpha} - 2i\cot a_1l} \quad \text{.....................(5),}$$

$$\frac{\phi_1}{\phi'} = \frac{2e^{ial}}{2\cos a_1l + i\sin a_1l\left(\alpha+\dfrac{1}{\alpha}\right)} \quad \text{..............(6).}$$

In order to pass to real quantities, these expressions must be put into the form $Re^{i\theta}$. *If* a_1 *be real,* we find corresponding to the incident wave

$$\phi = \cos(ax+by+ct),$$

the reflected wave

$$\phi = \frac{\left(\frac{1}{\alpha} - \alpha\right) \sin\left(-ax + by + ct - \epsilon\right)}{\sqrt{4 \cot^2 a_1 l + \left(\alpha + \frac{1}{\alpha}\right)^2}} \quad \text{.............(7)},$$

and the transmitted wave

$$\phi = \frac{2 \cos\left(ax + by + ct + al - \epsilon\right)}{\sqrt{4 \cos^2 a_1 l + \sin^2 a_1 l \left(\alpha + \frac{1}{\alpha}\right)^2}} \quad \text{.............(8)},$$

where

$$\tan \epsilon = \tfrac{1}{2}\left(\alpha + \frac{1}{\alpha}\right) \tan a_1 l \quad \text{.................(9)}.$$

If $\alpha = \frac{\cot\theta}{\cot\theta_1} \cdot \frac{\rho_1}{\rho} = 1$, there is no reflected wave, and the transmitted wave is represented by

$$\phi = \cos\left(ax + by + ct + al - a_1 l\right),$$

shewing that, except for the alteration of phase, the whole of the medium might as well have been uniform.

If l be small, we have approximately for the reflected wave

$$\phi = \frac{a_1 l}{2}\left(\frac{1}{\alpha} - \alpha\right) \sin\left(-ax + by + ct\right),$$

a formula applying when the plate is thin in comparison with the wave-length. Since $a_1 = \frac{2\pi}{\lambda_1} \cos\theta_1$, it appears that for a given angle of incidence the amplitude varies inversely as λ_1, or as λ.

In any case the reflection vanishes, if $\cot^2 a_1 l = \infty$, that is, if

$$2l \cos\theta_1 = m\lambda_1,$$

m being an integer. The wave is then wholly transmitted.

At perpendicular incidence, the intensity of the reflection is expressed by

$$\left(\frac{V\rho}{V_1\rho_1} - \frac{V_1\rho_1}{V\rho}\right) \div \sqrt{4 \cot^2 \frac{2\pi l}{V_1\tau} + \left(\frac{V\rho}{V_1\rho_1} + \frac{V_1\rho_1}{V\rho}\right)^2} \quad \text{.........(10)}.$$

Let us now suppose that the second medium is incompressible, so

that $V_1 = \infty$; our expression becomes

$$-\frac{\pi\frac{\rho_1 l}{\rho\lambda}}{\sqrt{1+\pi^2\left(\frac{\rho_1 l}{\rho\lambda}\right)^2}} \text{.......................(11)},$$

shewing how the amount of reflection depends upon the relative masses of such quantities of the media as have volumes in the ratio of $l:\lambda$. It is obvious that the second medium behaves like a rigid body and acts only in virtue of its inertia. If this be sufficient, the reflection may become sensibly total.

We have now to consider the case in which a_1 is imaginary. In the symbolical expressions (5) and (6) $\cos a_1 l$ and $i \sin a_1 l$ are real, while α, $\alpha+\frac{1}{\alpha}$, $\alpha-\frac{1}{\alpha}$ are pure imaginaries. Thus, if we suppose that $a_1 = ia_1'$, $\alpha = i\alpha'$, and introduce the notation of the hyperbolic sine and cosine (§ 170), we get

$$\frac{\phi''}{\phi'} = \frac{-i\left(\alpha'+\frac{1}{\alpha'}\right)\sinh a_1'l}{2\cosh a_1'l - i\left(\alpha'-\frac{1}{\alpha'}\right)\sinh a_1'l},$$

$$\frac{\phi_1}{\phi'} = \frac{2e^{ial}}{2\cosh a_1'l - i\left(\alpha'-\frac{1}{\alpha'}\right)\sinh a_1'l}.$$

Hence, if the incident wave be

$$\phi = \cos(ax+by+ct),$$

the reflected wave is expressed by

$$\phi = \frac{\left(\alpha'+\frac{1}{\alpha'}\right)\sinh a_1'l\cos(-ax+by+ct+\epsilon)}{\sqrt{4\cosh^2 a_1'l+\left(\alpha'-\frac{1}{\alpha'}\right)^2\sinh^2 a_1'l}} \text{.........(12)},$$

where

$$\cot\epsilon = \tfrac{1}{2}\left(\frac{1}{\alpha'}-\alpha'\right)\tanh a_1'l \text{...........(13)},$$

and the transmitted wave is expressed by

$$\phi = \frac{2\sin(ax+by+ct+al+\epsilon)}{\sqrt{4\cosh^2 a_1'l+\left(\alpha'-\frac{1}{\alpha'}\right)^2\sinh^2 a_1'l}} \text{............(14)}.$$

It is easy to verify that the energies of the reflected and transmitted waves account for the whole energy of the incident wave. Since in the present case the corresponding areas of wave-front are equal for all three waves, it is only necessary to add the squares of the amplitudes given in equations (7), (8), or in equations (12), (14).

272. These calculations of reflection and refraction under various circumstances might be carried further, but their interest would be rather optical than acoustical. It is important to bear in mind that no energy is destroyed by any number of reflections and refractions, whether partial or total, what is lost in one direction always reappearing in another.

On account of the great difference of densities reflection is usually nearly total at the boundary between air and any solid or liquid matter. Sounds produced in air are not easily communicated to water, and *vice versâ* sounds, whose origin is under water, are heard with difficulty in air. A beam of wood, or a metallic wire, acts like a speaking tube, conveying sounds to considerable distances with very little loss.

CHAPTER XIV.

GENERAL EQUATIONS.

273. In connection with the general problem of aërial vibrations in three dimensions one of the first questions, which naturally offers itself, is the determination of the motion in an unlimited atmosphere consequent upon arbitrary initial disturbances. It will be assumed that the disturbance is *small*, so that the ordinary approximate equations are applicable, and further that the initial velocities are such as can be derived from a velocity-potential, or (§ 240) that there is no *circulation*. If the latter condition be violated, the problem is one of vortex motion, on which we do not enter. We shall also suppose in the first place that no external forces act upon the fluid, so that the motion to be investigated is due solely to a disturbance actually existing at a time ($t=0$), previous to which we do not push our inquiries. The method that we shall employ is not very different from that of Poisson[1], by whom the problem was first successfully attacked.

If u_0, v_0, w_0 be the initial velocities at the point x, y, z, and s_0 the initial condensation, we have (§ 244),

$$\phi_0 = \int (u_0 dx + v_0 dy + w_0 dz) \quad\text{..................}(1),$$

$$\dot{\phi}_0 = -a^2 s_0 \quad\text{....................................}(2),$$

by which the initial values of the velocity-potential ϕ and of its differential coefficient with respect to time $\dot{\phi}$ are determined.

[1] Sur l'intégration de quelques équations linéaires aux différences partielles, et particulièrement de l'équation générale du mouvement des fluides élastiques. *Mém. de l'Institut*, t. III. p. 121. 1820.

The problem before us is to determine ϕ at time t from the above initial values, and the general equation applicable at all times and places,

$$\left(\frac{d^2}{dt^2} - a^2\nabla^2\right)\phi = 0 \quad\text{.......................} \quad (3).$$

When ϕ is known, its derivatives give the component velocities at any point.

The symbolical solution of (3) may be written

$$\phi = \sin(ia\nabla t).\theta + \cos(ia\nabla t).\chi \quad\text{...............}\quad (4),$$

where θ and χ are two arbitrary functions of x, y, z and $i = \sqrt{-1}$. To connect θ and χ with the initial values of ϕ and $\dot\phi$, which we shall denote by f and F respectively, it is only necessary to observe that when $t = 0$, (4) gives

$$\phi_0 = \chi, \quad \dot\phi_0 = ia\nabla.\theta;$$

so that our result may be expressed

$$\phi = \cos(ia\nabla t).f + \frac{\sin(ia\nabla t)}{ia\nabla}.F \quad\text{..............}\quad (5),$$

in which equation the question of the interpretation of odd powers of ∇ need not be considered, as both the symbolic functions are wholly even.

In the case where ϕ was a function of x only, we saw (§ 245) that its value for any point x at time t depended on the initial values of ϕ and $\dot\phi$ at the points whose co-ordinates were $x - at$ and $x + at$, and was wholly independent of the initial circumstances at all other points. In the present case the simplest supposition open to us is that the value of ϕ at a point O depends on the initial values of ϕ and $\dot\phi$ at points situated on the surface of the sphere, whose centre is O and radius at; and, as there can be no reason for giving one direction a preference over another, we are thus led to investigate the expression for the mean value of a function over a spherical surface in terms of the successive differential coefficients of the function at the centre.

By the symbolical form of Maclaurin's theorem the value of $F(x, y, z)$ at any point P on the surface of the sphere of radius r may be written

$$F(x, y, z) = e^{x\frac{d}{dx_0} + y\frac{d}{dy_0} + z\frac{d}{dz_0}} \cdot F(x_0, y_0, z_0),$$

the centre of the sphere O being the origin of co-ordinates. In the integration over the surface of the sphere $\frac{d}{dx_0}$, $\frac{d}{dy_0}$, $\frac{d}{dz_0}$ behave as constants; we may denote them temporarily by l, m, n, so that $\nabla^2 = l^2 + m^2 + n^2$.

Thus, r being the radius of the sphere, and dS an element of its surface, since, by the symmetry of the sphere, we may replace any function of $\frac{lx + my + nz}{\sqrt{(l^2 + m^2 + n^2)}}$ by the same function of z without altering the result of the integration,

$$\iint e^{lx+my+nz}\, dS = \iint (e^{\nabla})^{\frac{lx+my+nz}{\sqrt{(l^2+m^2+n^2)}}}\, dS$$

$$= \iint e^{\nabla z}\, dS = 2\pi r \int_{-r}^{+r} e^{\nabla z}\, dz = \frac{2\pi r}{\nabla}\left(e^{\nabla r} - e^{-\nabla r}\right) = 4\pi r^2 \frac{\sin(i\nabla r)}{i\nabla r}.$$

The mean value of F over the surface of the sphere of radius r is thus expressed by the result of the operation on F of the symbol $\frac{\sin(i\nabla r)}{i\nabla r}$, or, if $\iint d\sigma$ denote integration with respect to angular space,

$$\frac{1}{4\pi}\iint F(r)\, d\sigma = \frac{\sin(i\nabla r)}{i\nabla r} \cdot F \text{ .... } (6).$$

By comparison with (5) we now see that so far as ϕ depends on the initial values of $\dot{\phi}$, it is expressed by

$$\phi = \frac{t}{4\pi}\iint F(at)\, d\sigma \text{ .... } (7),$$

or in words, ϕ at any point at time t is the mean of the initial values of $\dot{\phi}$ over the surface of the sphere described round the point in question with radius at, the whole multiplied by t.

By Stokes' rule (§ 95), or by simple inspection of (5), we see that the part of ϕ depending on the initial values of ϕ may be derived from that just written by differentiating with respect to t and changing the arbitrary function. The complete value of ϕ at time t is therefore

$$\phi = \frac{t}{4\pi}\iint F(at)\,d\sigma + \frac{1}{4\pi}\frac{d}{dt}\,t\iint f(at)\,d\sigma \;\ldots\ldots\ldots(8),$$

which is Poisson's result[1].

On account of the importance of the present problem, it may be well to verify the solution *a posteriori*. We have first to prove that it satisfies the general differential equation (3). Taking for the present the first term only, and bearing in mind the general symbolic equation

$$\frac{d^2}{dt^2}\,t = \frac{1}{t}\frac{d}{dt}\,t^2\frac{d}{dt} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

we find from (8)

$$\frac{d^2\phi}{dt^2} = \frac{1}{4\pi t}\frac{d}{dt}\,t^2\iint\frac{d}{dt}F(at)\,d\sigma = \frac{1}{4\pi at}\frac{d}{dt}\iint\frac{dF(at)}{d(at)}\,dS,$$

dS being the surface element of the sphere $r = at$.

But by Green's theorem

$$\iint\frac{dF(r)}{dr}\,dS = \iiint\nabla^2F\,dV \quad (r < at);$$

and thus

$$\frac{d^2\phi}{dt^2} = \frac{1}{4\pi at}\frac{d}{dt}\cdot\iiint\nabla^2F\,dV \;(r < at)$$

$$= \frac{1}{4\pi t}\iint\nabla^2F\,dS \;(r = at) = \frac{a^2t}{4\pi}\iint\nabla^2F\,d\sigma.$$

Now $\iint\nabla^2F\,d\sigma$ is the same as $\nabla^2\iint F\,d\sigma$, and thus (3) is in fact satisfied.

Since the second part of ϕ is obtained from the first by differentiation, it also must satisfy the fundamental equation.

With respect to the initial conditions we see that when t is made equal to zero in (8),

$$\phi = \frac{1}{4\pi}\iint f(at)\,d\sigma \;(t=0) = f(0);$$

$$\dot\phi = \frac{1}{4\pi}\iint F(at)\,d\sigma \;(t=0) + \frac{1}{4\pi}\frac{d^2}{dt^2}\,t\iint f(at)\,d\sigma \;(t=0),$$

[1] Another investigation will be found in Kirchhoff's *Vorlesungen über Mathematische Physik*, p. 317. 1876.

of which the first term becomes in the limit $F(0)$. When $t=0$,

$$\frac{d^2}{dt^2}\, t\iint f(at)\, d\sigma = 2\,\frac{d}{dt}\iint f(at)\, d\sigma \quad (t=0)$$

$$= 2a\iint f'(at)\, d\sigma \quad (t=0) = 0,$$

since the oppositely situated elements cancel in the limit, when the radius of the spherical surface is indefinitely diminished. The expression in (8) therefore satisfies the prescribed initial conditions as well as the general differential equation.

274. If the initial disturbance be confined to a space T, the integrals in (8) § 273 are zero, unless some part of the surface of the sphere $r=at$ be included within T. Let O be a point external to T, r_1 and r_2 the radii of the least and greatest spheres described about O which cut it. Then so long as $at < r_1$, ϕ remains equal to zero. When at lies between r_1 and r_2, ϕ may be finite, but for values greater than r_2 ϕ is again zero. The disturbance is thus at any moment confined to those parts of space for which at is intermediate between r_1 and r_2. The limit of the wave is the envelope of spheres with radius at, whose centres are situated on the surface of T. "When t is small, this system of spheres will have an exterior envelope of two sheets, the outer of these sheets being exterior, and the inner interior to the shell formed by the assemblage of the spheres. The outer sheet forms the outer limit to the portion of the medium in which the dilatation is different from zero. As t increases, the inner sheet contracts, and at last its opposite sides cross, and it changes its character from being exterior, with reference to the spheres, to interior. It then expands, and forms the inner boundary of the shell in which the wave of condensation is comprised[1]." The successive positions of the boundaries of the wave are thus a series of parallel surfaces, and each boundary is propagated normally with a velocity equal to a.

If at the time $t=0$ there be no motion, so that the initial disturbance consists merely in a variation of density, the subsequent condition of things is expressed by the first term of (8) § 273. Let us suppose that the original disturbance, still limited to a finite region T, consists of condensation only, without rarefaction. It might be thought that the same peculiarity would attach to the

[1] Stokes, "Dynamical Theory of Diffraction," *Camb. Trans.* IX. p. 15.

resulting wave throughout the whole of its subsequent course; but, as Prof. Stokes has remarked, such a conclusion would be erroneous. For values of the time less than $r_1 \div a$ the potential at O is zero; it then becomes negative (s_0 being positive), and continues negative until it vanishes again when $t = r_2 \div a$, after which it always remains equal to zero. While ϕ is diminishing, the medium at O is in a state of condensation, but as ϕ increases again to zero, the state of the medium at O is one of rarefaction. The wave propagated outwards consists therefore of two parts at least, of which the first is condensed and the last rarefied. Whatever may be the character of the original disturbance within T, the final value of ϕ at any external point O is the same as the initial value, and therefore, since $a^2 s = -\dot{\phi}$, the mean condensation during the passage of the wave, depending on the integral $\int s\, dt$, is zero. Under the head of spherical waves we shall have occasion to return to this subject (§ 279).

The general solution embodied in (8) § 273 must of course embrace the particular case of plane waves, but a few words on this application may not be superfluous, for it might appear at first sight that the effect at a given point of a disturbance initially confined to a slice of the medium enclosed between two parallel planes would not pass off in any finite time, as we know it ought to do. Let us suppose for simplicity that ϕ_0 is zero throughout, and that within the slice in question the initial value $\dot{\phi}_0$ is constant. From the theory of plane waves we know that at any arbitrary point the disturbance will finally cease after the lapse of a time t, such that at is equal to the distance (d) of the point under consideration from the further boundary of the initially disturbed region; while on the other hand, since the sphere of radius at continues to cut the region, it would appear from the general formula that the disturbance continues. It is true indeed that ϕ remains finite, but this is not inconsistent with rest. It will in fact appear on examination that the mean value of $\dot{\phi}_0$ multiplied by the radius of the sphere is the same whatever may be the position and size of the sphere, provided only that it cut completely through the region of original disturbance. If $at > d$, ϕ is thus constant with respect both to space and time, and accordingly the medium is at rest.

275. In two dimensions, when ϕ is independent of z, it might be supposed that the corresponding formula would be obtained by

simply substituting for the sphere of radius at the circle of equal radius. This, however, is not the case. It may be proved that the mean value of a function $F(x, y)$ over the circumference of a circle of radius r is $J_0(ir\nabla)F_0$, where $i=\sqrt{-1}$,

$$\nabla^2 = \frac{d^2}{dx_0^2} + \frac{d^2}{dy_0^2},$$

and J_0 is *Bessel's* function of zero order; so that

$$\frac{1}{2\pi r}\int F(x,y)\,ds = \left(1 + \frac{r^2\nabla^2}{2^2} + \frac{r^4\nabla^4}{2^2 \cdot 4^2} + \dots\right)F,$$

differing from what is required to satisfy the fundamental equation.

The correct result applicable to two dimensions may be obtained from the general formula. The element of spherical surface dS may be replaced by $\frac{r\,dr\,d\theta}{\cos\psi}$, where r, θ are plane polar co-ordinates, and ψ is the angle between the tangent plane and that in which the motion takes place. Thus

$$\cos\psi = \frac{\sqrt{(a^2t^2 - r^2)}}{a\,t},$$

$F(at)$ is replaced by $F(r, \theta)$, and so

$$\phi = \iint \frac{F(r,\theta)\,r\,dr\,d\theta}{4\pi t\sqrt{a^2t^2 - r^2}} \quad\text{.......................(1)},$$

where the integration extends over the area of the circle $r = at$. The other term might be obtained by Stokes' rule.

This solution is applicable to the motion of a layer of gas between two parallel planes, or to that of an unlimited stretched membrane, which depends upon the same fundamental equation.

276. From the solution in terms of initial conditions we may, as usual (§ 66), deduce the effect of a continually renewed disturbance. Let us suppose that throughout the space T (which will ultimately be made to vanish), a uniform disturbance $\dot{\phi}$, equal to $\Phi(t')\,dt'$, is communicated at time t'. The resulting value of ϕ at time t is

$$\frac{S}{4\pi a^2(t-t')}\,\dot{\Phi}(t')\,dt',$$

where S denotes the part of the surface of the sphere $r = a(t - t')$

intercepted within T, a quantity which vanishes, unless $a(t-t')$ be compressed between the narrow limits r_1 and r_2. Ultimately $t-t'$ may be replaced by $r \div a$, and $\Phi(t')$ by $\Phi\left(t-\frac{r}{a}\right)$; and the result of the integration with respect to dt' is found by writing T (the volume) for $\int aS\,dt'$. Hence

$$\phi = \frac{T}{4\pi a^2 r}\,\Phi\left(t-\frac{r}{a}\right)\text{.......................}(1),$$

showing that the disturbance originating at any point spreads itself symmetrically in all directions with velocity a, and with amplitude varying inversely as the distance. Since any number of particular solutions may be superposed, the general solution of the equation

$$\ddot{\phi} = a^2\nabla^2\phi + \Phi\text{.......................}(2)$$

may be written

$$\phi = \frac{1}{4\pi a^2}\iiint \Phi\left(t-\frac{r}{a}\right)\frac{dV}{r}\text{..................}(3),$$

r denoting the distance of the element dV situated at x, y, z from O (at which ϕ is estimated), and $\Phi\left(t-\frac{r}{a}\right)$ the value of Φ for the point x, y, z at the time $t-\frac{r}{a}$. Complementary terms, satisfying through all space the equation $\ddot{\phi} = a^2\nabla^2\phi$, may of course occur independently.

In our previous notation (§ 244)

$$\Phi = \frac{d}{dt}\int (Xdx + Ydy + Zdz)\,;$$

and it is assumed that $Xdx+Ydy+Zdz$ is a complete differential. Forces, under whose action the medium could not adjust itself to equilibrium, are excluded; as for instance, a force uniform in magnitude and direction within a space T, and vanishing outside that space. The nature of the disturbance denoted by Φ is perhaps best seen by considering the extreme case when Φ vanishes except through a small volume, which is supposed to diminish without limit, while the magnitude of Φ increases in such a manner that the whole effect remains finite. If then we integrate equation (2)

through a small space including the point at which Φ is ultimately concentrated, we find in the limit

$$0 = a^2 \iint \frac{d\phi}{dn}\, dS + \iiint \Phi\, dV \text{..................(4)},$$

shewing that the effect of Φ may be represented by a proportional introduction or abstraction of fluid at the place in question. The simplest source of sound is thus analogous to a focus in the theory of conduction of heat, or to an electrode in the theory of electricity.

277. The preceding expressions are general in respect of the relation to time of the functions concerned; but in almost all the applications that we shall have to make, it will be convenient to analyse the motion by Fourier's theorem and treat separately the simple harmonic motions of various periods, afterwards, if necessary, compounding the results. The value of ϕ, and Φ, if simple harmonic at every point of space, may be expressed in the form $R\cos(nt+\epsilon)$, R and ϵ being independent of time, but variable from point to point. But as in such cases it often conduces to simplicity to add the term $iR\sin(nt+\epsilon)$, making altogether $Re^{i(nt+\epsilon)}$, or $Re^{i\epsilon}\,.\,e^{int}$, we will assume simply that all the functions which enter into a problem are proportional to e^{int}, the coefficients being in general complex. After our operations are completed, the real and imaginary parts of the expressions can be separated, either of them by itself constituting a solution of the question.

Since ϕ is proportional to e^{int}, $\ddot{\phi} = -n^2\phi$; and the differential equation becomes

$$\nabla^2\phi + \kappa^2\phi + a^{-2}\Phi = 0 \text{ ....(1)},$$

where, for the sake of brevity, κ is written in place of $n \div a$. If λ denote the *wave-length* of the vibration of the period in question,

$$\kappa = \frac{n}{a} = \frac{2\pi}{\lambda} \text{ ....(2)}.$$

To adapt (3) of the preceding section to the present case, it is only necessary to remark that the substitution of $t - \frac{r}{a}$ for t is effected by introducing the factor $e^{-in\frac{r}{a}}$, or $e^{-i\kappa r}$: thus

$$\Phi\left(t - \frac{r}{a}\right) = e^{-i\kappa r}\,\Phi(t),$$

and the solution of (1) is

$$\phi = \frac{1}{4\pi a^2}\iiint \frac{e^{-i\kappa r}}{r}\,\Phi\,dV \quad\text{.................(3),}$$

to which may be added any solution of $\nabla^2\phi + \kappa^2\phi = 0$.

If the disturbing forces be all in the same phase, and the region through which they act be very small in comparison with the wave-length, $e^{-i\kappa r}$ may be removed from under the integral sign, and at a sufficient distance we may take

$$\phi = \frac{e^{-i\kappa r}}{4\pi a^2 r}\iiint \Phi\,dV,$$

or in real quantities, on restoring the time factor and replacing $\iiint \Phi\,dV$ by Φ_1,

$$\phi = \Phi_1 \frac{\cos(nt - \kappa r + \epsilon)}{4\pi a^2 r} \quad\text{.................... (4).}$$

In order to verify that (3) satisfies the differential equation (1), we may proceed as in the theory of the common potential. Considering one element of the integral at a time, we have first to shew that

$$\phi = \frac{e^{-i\kappa r}}{r} \quad\text{.......................... (5)}$$

satisfies $\nabla^2\phi + \kappa^2\phi = 0$, at points for which r is finite. The simplest course is to express ∇^2 in polar co-ordinates referred to the element itself as pole, when it appears that

$$\nabla^2 \frac{e^{-i\kappa r}}{r} = \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\frac{e^{-i\kappa r}}{r} = \frac{1}{r}\frac{d^2}{dr^2} r \,.\, \frac{e^{-i\kappa r}}{r} = -\kappa^2\,\frac{e^{-i\kappa r}}{r}.$$

We infer that (3) satisfies $\nabla^2\phi + \kappa^2\phi = 0$, at all points for which Φ vanishes. In the case of a point at which Φ does not vanish, we may put out of account all the elements situated at a finite distance (as contributing only terms satisfying $\nabla^2\phi + \kappa^2\phi = 0$), and for the element at an infinitesimal distance replace $e^{-i\kappa r}$ by unity. Thus on the whole

$$(\nabla^2 + \kappa^2)\,\phi = \frac{1}{4\pi a^2}\nabla^2 \iiint \Phi\frac{dV}{r} = \frac{1}{a^2}\Phi,$$

exactly as in Poisson's theorem for the common potential[1].

[1] See Thomson and Tait's *Nat. Phil.*, § 491.

278. The effect of a force Φ_1 distributed over a surface S may be obtained as a limiting case from (3) § 277. $\Phi\, dV$ is replaced by $\Phi\, b dS$, b denoting the thickness of the layer; and in the limit we may write $\Phi\, b = \Phi_1$. Thus

$$\phi = \frac{1}{4\pi a^2}\iint \Phi_1 \frac{e^{-ikr}}{r}\, dS \qquad (1).$$

The value of ϕ is the same on the two sides of S, but there is discontinuity in its derivatives. If dn be drawn outwards from S normally, (4) § 276 gives

$$\left(\frac{d\phi}{dn}\right)_1 + \left(\frac{d\phi}{dn}\right)_2 = -\frac{1}{a^2}\Phi_1 \qquad (2)^1.$$

If the surface S be plane, the integral in (1) is evidently symmetrical with respect to it, and therefore

$$\left(\frac{d\phi}{dn}\right)_1 = \left(\frac{d\phi}{dn}\right)_2.$$

Hence, if $\frac{d\phi}{dn}$ be the given normal velocity of the fluid in contact with the plane, the value of ϕ is determined by

$$\phi = -\frac{1}{2\pi}\iint \frac{d\phi}{dn}\frac{e^{-ikr}}{r}\, dS \qquad (3),$$

which is a result of considerable importance. To exhibit it in terms of real quantities, we may take

$$\frac{d\phi}{dn} = P e^{i(nt+\epsilon)} \qquad (4),$$

P and ϵ being real functions of the position of dS. The symbolical solution then becomes

$$\phi = -\frac{1}{2\pi}\iint P\, e^{i(nt-kr+\epsilon)}\frac{dS}{r} \qquad (5),$$

from which, if the imaginary part be rejected, we obtain

$$\phi = -\frac{1}{2\pi}\iint P\,\frac{\cos(nt-kr+\epsilon)}{r}\, dS \qquad (6),$$

corresponding to

$$\frac{d\phi}{dn} = P\cos(nt+\epsilon) \qquad (7).$$

[1] Helmholtz. *Crelle*, t. 57, p. 21.

The same method is applicable to the general case when the motion is not restricted to be simple harmonic. We have

$$\phi = -\frac{1}{2\pi}\iint V\left(t-\frac{r}{a}\right)\cdot\frac{dS}{r} \quad\text{..................}\quad (8),$$

where by $V\left(t-\frac{r}{a}\right)$ is denoted the normal velocity at the plane for the element dS at the time $t-(r \div a)$, that is to say, at a time $r \div a$ antecedent to that at which ϕ is estimated.

In order to complete the solution of the problem for the unlimited mass of fluid lying on one side of an infinite plane, we have to add the most general value of ϕ, consistent with $V=0$. This part of the question is identical with the general problem of reflection from an infinite rigid plane[1].

It is evident that the effect of the constraint will be represented by the introduction on the other side of the plane of fictitious initial displacements and forces, forming in conjunction with those actually existing on the first side a system perfectly symmetrical with respect to the plane. Whatever the initial values of ϕ and $\dot{\phi}$ may be belonging to any point on the first side, the same must be ascribed to its *image*, and in like manner whatever function of the time Φ may be at the first point, it must be conceived to be the same function of the time at the other. Under these circumstances it is clear that for all future time ϕ will be symmetrical with respect to the plane, and therefore the normal velocity zero. So far then as the motion on the first side is concerned, there will be no change if the plane be removed, and the fluid continued indefinitely in all directions, provided the circumstances on the second side are the exact reflection of those on the first. This being understood, the general solution of the problem for a fluid bounded by an infinite plane is contained in the formulæ (8) § 273, (3) § 277, and (8) of the present section. They give the result of arbitrary initial conditions (ϕ_0 and $\dot{\phi}_0$), arbitrary applied forces (Φ), and arbitrary motion of the plane (V).

Measured by the resulting potential, a source of given magnitude, i.e. a source at which a given introduction and withdrawal of fluid takes place, is thus twice as effective when close to a rigid plane, as if it were situated in the open; and the result is ulti-

[1] Poisson, *Journal de l'école polytechnique*, t. VII. 1808.

mately the same, whether the source be concentrated in a point close to the plane, or be due to a corresponding normal motion of the surface of the plane itself.

The operation of the plane is to double the effective pressures which oppose the expansion and contraction at the source, and therefore to double the total energy emitted; and since this energy is diffused through only the half of angular space, the intensity of the sound is quadrupled, which corresponds to a doubled amplitude, or potential (§ 245).

We will now suppose that instead of $\frac{d\phi}{dn} = 0$, the prescribed condition at the infinite plane is that $\phi = 0$. In this case the fictitious distribution of ϕ_0, $\dot{\phi}_0$, Φ, on the second side of the plane must be the *opposite* of that on the first side, so that the sum of the values at two corresponding points is always zero. This secures that on the plane of symmetry itself ϕ shall vanish throughout.

Let us next suppose that there are two parallel surfaces S_1, S_2, separated by the infinitely small interval dn, and that the value of Φ_1 on the second surface is equal and opposite to the value of Φ_1 on the first. In crossing S_1, there is by (2) a finite change in the value of $\frac{d\phi}{dn}$ to the amount of $\Phi_1 \div a^2$, but in crossing S_2 the same finite change occurs in the reverse direction. When dn is reduced without limit, and $\Phi_1 dn$ replaced by Φ_{11}, $\frac{d\phi}{dn}$ will be the same on the two sides of the double sheet, but there will be discontinuity in the value of ϕ to the amount of $\Phi_{11} \div a^2$. At the same time (1) becomes

$$\phi = \frac{1}{4\pi a^2} \iint \frac{d}{dn}\left(\frac{e^{-ikr}}{r}\right) \Phi_{11}\, dS \text{..................} (9).$$

If the surface S be plane, the values of ϕ on the two sides of it are numerically equal, and therefore close to the surface itself

$$\phi = \pm \tfrac{1}{2} a^{-2}\, \Phi_{11}.$$

Hence (9) may be written

$$\phi = -\frac{1}{2\pi} \iint \frac{d}{dn}\left(\frac{e^{-ikr}}{r}\right) \phi\, dS \text{..................} (10),$$

where ϕ under the integral sign represents the surface-potential, positive on the one side and negative on the other, due to the

action of the forces at S. The direction of dn must be understood to be *towards* the side at which ϕ is to be estimated.

279. The problem of spherical waves diverging from a point has already been forced upon us and in some degree considered, but on account of its importance it demands a more detailed treatment. If the centre of symmetry be taken as pole the velocity-potential is a function of r only, and (§ 241) ∇^2 reduces to $\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}$, or to $\frac{1}{r}\frac{d^2}{dr^2}r$. The equation of free motion (3) § 273 thus becomes

$$\frac{d^2(r\phi)}{dt^2}=a^2\frac{d^2(r\phi)}{dr^2}\text{..........................(1)},$$

whence, as in § 245,

$$r\phi=f(at-r)+F(at+r)\text{..................(2)}.$$

The values of the velocity and condensation are to be found by differentiation in accordance with the formulæ

$$u=\frac{d\phi}{dr},\qquad s=-\frac{1}{a^2}\frac{d\phi}{dt}\text{......................(3)}.$$

As in the case of one dimension, the first term represents a wave advancing in the direction of r increasing, that is to say, a divergent wave, and the second term represents a wave converging upon the pole. The latter does not in itself possess much interest. If we confine our attention to the divergent wave, we have

$$u=-\frac{f(at-r)}{r^2}-\frac{f'(at-r)}{r};\qquad as=-\frac{f'(at-r)}{r}\text{......(4)}.$$

When r is very great the term divided by r^2 may be neglected, and then approximately

$$u=as\text{..................................(5)},$$

the same relation as obtains in the case of a plane wave, as might have been expected.

If the type be harmonic,

$$r\phi=A\,e^{i\kappa(at-r+\theta)}\text{..........................(6)},$$

or, if only the real part be retained,

$$r\phi=A\cos\frac{2\pi}{\lambda}(at+\theta-r)\text{..................(7)}.$$

If a divergent disturbance be confined to a spherical shell, within and without which there is neither condensation nor velocity, the character of the wave is limited by a remarkable relation, first pointed out by Stokes[1]. From equations (4) we have

$$(as - u)\, r^2 = f(at - r),$$

shewing that the value of $f(at-r)$ is the same, viz. zero, both inside and outside the shell to which the wave is limited. Hence by (4), if α and β be radii less and greater than the extreme radii of the shell,

$$\int_\alpha^\beta s\, r\, dr = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

which is the expression of the relation referred to. As in § 274, we see that a condensed or a rarefied wave cannot exist alone. When the radius becomes great in comparison with the thickness, the variation of r in the integral may be neglected, and (8) then expresses that the *mean* condensation is zero.

In applying the general solution (2) to deduce the motion resulting from arbitrary initial circumstances, we must remember that in its present form it is too general for the purpose, since it covers the case in which the pole is itself a source, or place where fluid is introduced or withdrawn in violation of the equation of continuity. The total current across the surface of a sphere of radius r is $4\pi r^2 u$, or by (2) and (3)

$$-4\pi \{f(at-r) + F(at+r)\} + 4\pi r\{F'(at+r) - f'(at-r)\},$$

so that, if the pole be not a source, $f(at-r) + F(at+r)$, or $r\phi$, must vanish with r. Thus

$$f(at) + F(at) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

an equation which must hold good for all positive values of the argument[2].

By the known initial circumstances the values of u and s are determined for the time $t = 0$, and for all (positive) values of r.

[1] *Phil. Mag.* XXXIV. p. 52. 1849.

[2] The solution for spherical vibrations may be obtained without the use of (1) by superposition of trains of plane waves, related similarly to the pole, and travelling outwards in all directions symmetrically.

If these initial values be represented by u_0 and s_0, we obtain from (2) and (3)

$$\left.\begin{array}{l} f(-r)+F(r)=r\int u_0\,dr \\ f(-r)-F(r)=a\int s_0\,r\,dr \end{array}\right\}\text{..................(10),}$$

by which the function f is determined for all negative arguments, and the function F for all positive arguments. The form of f for positive arguments follows by means of (9), and then the whole subsequent motion is determined by (2). The form of F for negative arguments is not required.

The initial disturbance divides itself into two parts, travelling in opposite directions, in each of which $r\phi$ is propagated with constant velocity a, and the inwards travelling wave is continually reflected at the pole. Since the condition to be there satisfied is $r\phi=0$, the case is somewhat similar to that of a parallel tube terminated by an *open* end, and we may thus perhaps better understand why the condensed wave, arising from the liberation of a mass of condensed air round the pole, is followed immediately by a wave of rarefaction.

280. Returning now to the case of a train of harmonic waves travelling outwards continually from the pole as source, let us investigate the connection between the velocity-potential and the quantity of fluid which must be supposed to be introduced and withdrawn alternately. If the velocity-potential be

$$\phi=-\frac{A}{4\pi r}\cos\kappa(at-r)\text{ .... (1),}$$

we have, as in the preceding section, for the total current crossing a sphere of radius r,

$$4\pi r^2\frac{d\phi}{dr}=A\{\cos\kappa(at-r)-\kappa r\sin\kappa(at-r)\}=A\cos\kappa at,$$

when r is small enough. If the maximum rate of introduction of fluid be denoted by A, the corresponding potential is given by (1).

It will be observed that when the source, as measured by A, is finite, the potential and the pressure-variation (proportional to $\dot{\phi}$) are infinite at the pole. But this does not, as might for a moment be supposed, imply an infinite emission of energy. If the pressure

be divided into two parts, one of which has the same phase as the velocity, and the other the same phase as the acceleration, it will be found that the former part, on which the work depends, is finite. The infinite part of the pressure does no work on the whole, but merely keeps up the vibration of the air immediately round the source, whose effective inertia is indefinitely great.

We will now investigate the energy emitted from a simple source of given magnitude, supposing for the sake of greater generality that the source is situated at the vertex of a rigid cone of solid angle ω. If the rate of introduction of fluid at the source be $A \cos \kappa a t$, we have

$$\omega r^2 \frac{d\phi}{dr} = A \cos \kappa a t$$

ultimately, corresponding to

$$\phi = -\frac{A}{\omega r} \cos \kappa (at - r) \qquad \text{(2)};$$

whence

$$\dot{\phi} = \frac{\kappa a A}{\omega r} \sin \kappa (at - r) \qquad \text{(3)},$$

and

$$\omega r^2 \frac{d\phi}{dr} = A \{\cos \kappa (at - r) - \kappa r \sin \kappa (at - r)\} \qquad \text{(4)}.$$

Thus, as in § 245, if dW be the work transmitted in time dt, we get, since $\delta p = -\rho \dot{\phi}$,

$$\frac{dW}{dt} = -\frac{\rho \kappa a A^2}{\omega r} \sin \kappa (at - r) \cos \kappa (at - r) + \rho \frac{\kappa^2 a A^2}{\omega} \sin^2 \kappa (at - r).$$

Of the right-hand member the first term is entirely periodic, and in the second the mean value of $\sin^2 \kappa (at - r)$ is $\frac{1}{2}$. Thus in the long run

$$W = \frac{\rho \kappa^2 a A^2}{2\omega} t \qquad \text{(5)[1]}.$$

It will be remarked that when the source is given, the amplitude varies inversely as ω, and therefore the intensity inversely as ω^2. For an acute cone the intensity is greater, not only on account of the diminution in the solid angle through which the

[1] Cambridge Mathematical Tripos Examination, 1876.

sound is distributed, but also because the total energy emitted from the source is itself increased.

When the source is in the open, we have only to put $\omega = 4\pi$, and when it is close to a rigid plane, $\omega = 2\pi$.

The results of this article find an interesting application in the theory of the speaking trumpet, or (by the law of reciprocity §§ 109, 294) hearing trumpet. If the diameter of the large open end be small in comparison with the wave-length, the waves on arrival suffer copious reflection, and the ultimate result, which must depend largely on the precise relative lengths of the tube and of the wave, requires to be determined by a different process. But by sufficiently prolonging the cone, this reflection may be diminished, and it will tend to cease when the diameter of the open end includes a large number of wave-lengths. Apart from friction it would therefore be possible by diminishing ω to obtain from a given source any desired amount of energy, and at the same time by lengthening the cone to secure the unimpeded transference of this energy from the tube to the surrounding air.

From the theory of diffraction it appears that the sound will not fall off to any great extent in a lateral direction, unless the diameter at the large end exceed half a wave-length. The ordinary explanation of the effect of a common trumpet, depending on a supposed concentration of *rays* in the axial direction, is thus untenable.

281. By means of Euler's equation,

$$\frac{d^2(r\phi)}{dt^2} = a^2 \frac{d^2(r\phi)}{dr^2} \quad \text{..........................} \quad (1),$$

we may easily establish a theory for conical pipes with open ends, analogous to that of Bernoulli for parallel tubes, subject to the same limitation as to the smallness of the diameter of the tubes in comparison with the wave-length of the sound. Assuming that the vibration is stationary, so that $r\phi$ is everywhere proportional to $\cos \kappa at$, we get from (1)

$$\frac{d^2(r\phi)}{dr^2} + \kappa^2 . r\phi = 0 \quad \text{........................} \quad (2),$$

of which the general solution is

$$r\phi = A \cos \kappa r + B \sin \kappa r \quad \text{....................} \quad (3).$$

The condition to be satisfied at an open end, viz., that there is to be no condensation or rarefaction, gives $r\phi = 0$, so that, if the extreme radii of the tube be r_1 and r_2, we have

$$A \cos \kappa r_1 + B \sin \kappa r_1 = 0, \quad A \cos \kappa r_2 + B \sin \kappa r_2 = 0,$$

whence by elimination of $A : B$, $\sin \kappa (r_2 - r_1) = 0$, or $r_2 - r_1 = \frac{1}{2} m\lambda$, where m is an integer. In fact since the form of the general solution (3) and the condition for an open end are the same as for a parallel tube, the result that the length of the tube is a multiple of the half wave-length is necessarily also the same.

A cone, which is complete as far as the vertex, may be treated as if the vertex were an open end, since, as we saw in § 279, the condition $r\phi = 0$ is there satisfied.

The resemblance to the case of parallel tubes does not extend to the position of the nodes. In the case of the gravest vibration of a parallel tube open at both ends, the node occupies a central position, and the two halves vibrate synchronously as tubes open at one end and stopped at the other. But if a conical tube were divided by a partition at its centre, the two parts would have different periods, as is evident, because the one part differs from a parallel tube by being contracted at its open end where the effect of a contraction is to depress the pitch, while the other part is contracted at its stopped end, where the effect is to raise the pitch. In order that the two periods may be the same, the partition must approach nearer to the narrower end of the tube. Its actual position may be determined analytically from (3) by equating to zero the value of $\frac{d\phi}{dr}$.

When both ends of a conical pipe are closed, the corresponding notes are determined by eliminating $A : B$ between the equations,

$$A (\cos \kappa r_1 + \kappa r_1 \sin \kappa r_1) + B (\sin \kappa r_1 - \kappa r_1 \cos \kappa r_1) = 0,$$

$$A (\cos \kappa r_2 + \kappa r_2 \sin \kappa r_2) + B (\sin \kappa r_2 - \kappa r_2 \cos \kappa r_2) = 0,$$

of which the result may be put into the form

$$\kappa r_2 - \tan^{-1} \kappa r_2 = \kappa r_1 - \tan^{-1} \kappa r_1 \quad \ldots\ldots\ldots\ldots\ldots (4).$$

If $r_1 = 0$, we have simply

$$\tan \kappa r_2 = \kappa r_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)^1;$$

[1] For the roots of this equation see § 207.

if r_1 and r_2 be very great, $\tan^{-1}\kappa r_1$ and $\tan^{-1}\kappa r_2$ are both odd multiples of $\frac{1}{2}\pi$, so that $r_2 - r_1$ is a multiple of $\frac{1}{2}\lambda$, as the theory of parallel tubes requires.

282. If there be two distinct sources of sound of the same pitch, situated at O_1 and O_2, the velocity-potential ϕ at a point P whose distances from O_1, O_2 are r_1 and r_2, may be expressed

$$\phi = A\frac{\cos\kappa(at - r_1)}{r_1} + B\frac{\cos\kappa(at - r_2 - \alpha)}{r_2} \quad\ldots\ldots\ldots(1),$$

where A and B are coefficients representing the magnitudes of the sources, (which without loss of generality may be supposed to have the same sign), and α represents the retardation (considered as a distance) of the second source relatively to the first. The two trains of spherical waves are in agreement at any point P, if $r_2 + \alpha - r_1 = \pm m\lambda$, where m is an integer, that is, if P lie on any one of a system of hyperboloids of revolution having foci at O_1 and O_2. At points lying on the intermediate hyperboloids, represented by $r_2 + \alpha - r_1 = \pm\frac{1}{2}(2m+1)\lambda$, the two sets of waves are opposed in phase, and neutralize one another as far as their actual magnitudes permit. The neutralization is complete, if $r_1 : r_2 = A : B$, and then the density at P continues permanently unchanged. The intersections of this sphere with the system of hyperboloids will thus mark out in most cases several circles of absolute silence. If the distance O_1O_2 between the sources be great in comparison with the length of a wave, and the sources themselves be not very unequal in power, it will be possible to depart from the sphere $r_1 : r_2 = A : B$ for a distance of several wave-lengths, without appreciably disturbing the equality of intensities, and thus to obtain over finite surfaces several alternations of sound and of almost complete silence.

There is some difficulty in actually realising a satisfactory interference of two independent sounds. Unless the unison be extraordinarily perfect, the silences are only momentary and are consequently difficult to appreciate. It is therefore best to employ sources which are mechanically connected in such a way that the relative phases of the sounds issuing from them cannot vary. The simplest plan is to repeat the first sound by reflection from a flat wall (§§ 269, 278), but the experiment then loses something in directness owing to the fictitious character of the second source. Perhaps the most satisfactory form of the experiment is that

described in the Philosophical Magazine for June 1877 by myself. "An intermittent electric current, obtained from a fork interrupter making 128 vibrations per second, excited by means of electro-magnets two other forks, whose frequency was 256, (§§ 63, 64). These latter forks were placed at a distance of about ten yards apart, and were provided with suitably tuned resonators, by which their sounds were reinforced. The pitch of the forks was necessarily identical, since the vibrations were forced by electro-magnetic forces of absolutely the same period. With one ear closed it was found possible to define the places of silence with considerable accuracy, a motion of about an inch being sufficient to produce a marked revival of sound. At a point of silence, from which the line joining the forks subtended an angle of about 60°, the apparent striking up of one fork, when the other was stopped, had a very peculiar effect."

Another method is to duplicate a sound coming along a tube by means of branch tubes, whose open ends act as sources. But the experiment in this form is not a very easy one.

It often happens that considerations of symmetry are sufficient to indicate the existence of places of silence. For example, it is evident that there can be no variation of density in the continuation of the plane of a vibrating plate, nor in the equatorial plane of a symmetrical solid of revolution vibrating in the direction of its axis. More generally, any plane is a plane of silence, with respect to which the sources are symmetrical in such a manner that at any point and at its image in the plane there are sources of equal intensities and of opposite phases, or, as it is often more conveniently expressed, of the same phase and of opposite amplitudes.

If any number of sources in the same phase, whose amplitudes are on the whole as much negative as positive, be placed on the circumference of a circle, they will give rise to no disturbance of pressure at points on the straight line which passes through the centre of the circle and is directed at right angles to its plane. This is the case of the symmetrical bell (§ 232), which emits no sound in the direction of its axis[1].

The accurate experimental investigation of aërial vibrations is beset with considerable difficulties, which have been only partially

[1] *Phil. Mag.* (5), III. p. 460. 1877.

surmounted hitherto. In order to avoid unwished for reflections it is generally necessary to work in the open air, where delicate apparatus, such as a sensitive flame, is difficult of management. Another impediment arises from the presence of the experimenter himself, whose person is large enough to disturb materially the state of things which he wishes to examine. Among indicators of sound may be mentioned membranes stretched over cups, the agitation being made apparent by sand, or by small pendulums resting lightly against them. If a membrane be simply stretched across a hoop, both its faces are acted upon by nearly the same forces, and consequently the motion is much diminished, unless the membrane be large enough to cast a sensible shadow, in which its hinder face may be protected. Probably the best method of examining the intensity of sound at any point in the air is to divert a portion of it by means of a tube ending in a small cone or resonator, the sound so diverted being led to the ear, or to a manometric capsule. In this way it is not difficult to determine places of silence with considerable precision.

By means of the same kind of apparatus it is possible to examine even the *phase* of the vibration at any point in air, and to trace out the surfaces on which the phase does not vary[1]. If the interior of a resonator be connected by flexible tubing with a manometric capsule, which influences a small gas flame, the motion of the flame is related in an invariable manner (depending on the apparatus itself) to the variation of pressure at the mouth of the resonator; and in particular the interval between the lowest drop of the flame and the lowest pressure at the resonator is independent of the absolute time at which these effects occur. In Mayer's experiment two flames were employed, placed close together in one vertical line, and were examined with a revolving mirror. So long as the associated resonators were undisturbed, the serrations of the two flames occupied a fixed relative position, and this relative position was also maintained when one resonator was moved about so as to trace out a surface of invariable phase. For further details the reader must be referred to the original paper.

283. When waves of sound impinge upon an obstacle, a portion of the motion is thrown back as an echo, and under cover of the obstacle there is formed a sort of sound shadow. In order, however, to produce shadows in anything like optical perfection,

[1] Mayer, *Phil. Mag.* (4), XLIV. p. 321. 1872.

the dimensions of the intervening body must be considerable. The standard of comparison proper to the subject is the wave-length of the vibration; it requires almost as extreme conditions to produce *rays* in the case of sound, as it requires in optics to avoid producing them. Still, sound shadows thrown by hills, or buildings, are often tolerably complete, and must be within the experience of all.

For closer examination let us take first the case of plane waves of harmonic type impinging upon an immovable plane screen, of infinitesimal thickness, in which there is an aperture of any form, the plane of the screen $(x=0)$ lying parallel to the fronts of the waves. The velocity-potential of the undisturbed train of waves may be taken,

$$\phi = \cos(nt - \kappa x) \qquad (1).$$

If the value of $\frac{d\phi}{dx}$ over the aperture be known, formulæ (6) and (7) § 278 allow us to calculate the value of ϕ at any point on the further side. In the ordinary theory of diffraction, as given in works on optics, it is assumed that the disturbance in the plane of the aperture is the same as if the screen were away. This hypothesis, though it can never be rigorously exact, will suffice when the aperture is very large in comparison with the wave-length, as is usually the case in optics.

For the undisturbed wave we have

$$\frac{d\phi}{dx}(x=0) = \kappa \sin nt \qquad (2),$$

and therefore on the further side, we get

$$\phi = -\frac{\kappa}{2\pi}\iint \frac{\sin(nt-\kappa r)}{r}\, dS \qquad (3),$$

the integration extending over the area of the aperture. Since $\kappa = 2\pi \div \lambda$, we see by comparison with (1) that in supposing a primary wave broken up, with the view of applying Huyghens' principle, dS must be divided by λr, and the phase must be accelerated by a quarter of a period.

When r is large in comparison with the dimensions of the aperture, the composition of the integral is best studied by the aid of Huyghens' zones. With the point O, for which ϕ is to be estimated, as centre describe a series of spheres of radii increasing

by the constant difference $\frac{1}{2}\lambda$, the first sphere of the series being of such radius (c) as to touch the plane of the screen. On this plane are thus marked out a series of circles, whose radii ρ are given by $\rho^2 + c^2 = (c + \frac{1}{2}n\lambda)^2$, or $\rho^2 = nc\lambda$, very nearly; so that the rings into which the plane is divided, being of approximately equal area, make contributions to ϕ which are approximately equal in numerical magnitude and alternately opposite in sign. If O lie decidedly within the projection of the area, the first term of the series representing the integral is finite, and the terms which follow are alternately opposite in sign and of numerical magnitude at first nearly constant, but afterwards diminishing gradually to zero, as the parts of the rings intercepted within the aperture become less and less. The case of an aperture, whose boundary is equidistant from O, is excepted.

In a series of this description any term after the first is neutralized almost exactly by half the sum of those which immediately precede and follow it, so that the sum of the whole series is represented approximately by half the first term, which stands over uncompensated. We see that, provided a sufficient number of zones be included within the aperture, the value of ϕ at the point O is independent of the nature of the aperture, and is therefore the same as if there had been no screen at all. Or we may calculate directly the effect of the circle with which the system of zones begins; a course which will have the advantage of bringing out more clearly the significance of the change of phase which we found it necessary to introduce when the primary wave was broken up. Thus, let us conceive the circle in question divided into infinitesimal rings of equal area. The parts of ϕ due to each of these rings are equal in amplitude and of phase ranging uniformly over half a complete period. The phase of the resultant is therefore midway between those of the extreme elements, that is to say, a quarter of a period behind that due to the element at the centre of the circle. The amplitude of the resultant will be less than if all its components had been in the same phase, in the ratio $\int_0^\pi \sin x\, dx : \pi$, or $2 : \pi$; and therefore since the area of the circle is $\pi\lambda c$, half the effect of the first zone is

$$\phi = -\tfrac{1}{2} \cdot \frac{2}{\pi} \cdot \frac{\sin(nt - kc - \frac{1}{2}\pi)}{\lambda c} \cdot \pi\lambda c = \cos(nt - \kappa c),$$

the same as if the primary wave were to pass on undisturbed.

When the point O is well away from the projection of the aperture, the result is quite different. The series representing the integral then converges at both ends, and by the same reasoning as before its sum is seen to be approximately zero. We conclude that if the projection of O on the plane $x=0$ fall within the aperture, and be nearer to O by a great many wave-lengths than the nearest point of the boundary of the aperture, then the disturbance at O is nearly the same as if there were no obstacle at all; but, if the projection of O fall outside the aperture and be nearer to O by a great many wave-lengths than the nearest point of the boundary, then the disturbance at O practically vanishes. This is the theory of sound rays in its simplest form.

The argument is not very different if the screen be oblique to the plane of the waves. As before, the motion on the further side of the screen may be regarded as due to the normal motion of the particles in the plane of the aperture, but this normal motion now varies in phase from point to point. If the primary waves proceed from a source at Q, Huyghens' zones for a point P are the series of ellipses represented by $r_1 + r_2 = PQ + \frac{1}{2}n\lambda$, where r_1 and r_2 are the distances of any point on the screen from Q and P respectively, and n is an integer. On account of the assumed smallness of λ in comparison with r_1 and r_2, the zones are at first of equal area and make equal and opposite contributions to the value of ϕ; and thus by the same reasoning as before we may conclude that at any point decidedly outside the geometrical projection of the aperture the disturbance vanishes, while at any point decidedly within the geometrical projection the disturbance is the same as if the primary wave had passed the screen unimpeded. It may be remarked that the increase of area of the Huyghens' zones due to obliquity is compensated in the calculation of the integral by the correspondingly diminished value of the normal velocity of the fluid. The enfeeblement of the primary wave between the screen and the point P due to divergency is represented by a diminution in the area of the Huyghens' zones below that corresponding to plane incident waves in the ratio $r_1 + r_2 : r_1$.

There is a simple relation between the transmission of sound through an aperture in a screen and its reflection from a plane reflector of the same form as the aperture, of which advantage may sometimes be taken in experiment. Let us imagine a source similar to Q and in the same phase to be placed at Q', the *image* of

Q in the plane of the screen, and let us suppose that the screen is removed and replaced by a plate whose form and position is exactly that of the aperture; then we know that the effect at P of the two sources is uninfluenced by the presence of the plate, so that the vibration from Q' reflected from the plate and the vibration from Q transmitted round the plate together make up the same vibration as would be received from Q if there were no obstacle at all. Now according to the assumption which we made at the beginning of this section, the unimpeded vibration from Q may be regarded as composed of the vibration that finds its way round the plate and of that which would pass an aperture of the same form in an infinite screen, and thus the vibration from Q as transmitted through the aperture is equal to the vibration from Q' as reflected from the plate.

In order to obtain a nearly complete reflection it is not necessary that the reflecting plate include more than a small number of Huyghens' zones. In the case of direct reflection the radius ρ of the first zone is determined by the equation

$$\rho^2\left(\frac{1}{c_1}+\frac{1}{c_2}\right)=\lambda \qquad\qquad (4),$$

where c_1 and c_2 are the distances from the reflector of the source and of the point of observation. When the distances concerned are great, the zones become so large that ordinary walls are insufficient to give a complete reflection, but at more moderate distances echos are often nearly perfect. The area necessary for complete reflection depends also upon the wave-length; and thus it happens that a board or plate, which would be quite inadequate to reflect a grave musical note, may reflect very fairly a hiss or the sound of a high whistle. In experiments on reflection by screens of moderate size, the principal difficulty is to get rid sufficiently of the direct sound. The simplest plan is to reflect the sound from an electric bell, or other fairly steady source, round the corner of a large building[1].

284. In the preceding section we have applied Huyghens' principle to the case where the primary wave is supposed to be broken up at the surface of an imaginary plane. If we really know what the normal motion at the plane is, we can calculate

[1] *Phil. Mag.* (5) III. p. 458. 1877.

the disturbance at any point on the further side by a rigorous process. For surfaces other than the plane the problem has not been solved generally; nevertheless, it is not difficult to see that when the radii of curvature of the surface are very great in comparison with the wave-length, the effect of a normal motion of an element of the surface must be very nearly the same as if the surface were plane. On this understanding we may employ the same integral as before to calculate the aggregate result. As a matter of convenience it is usually best to suppose the wave to be broken up at what is called in optics a *wave-surface*, that is, a surface at every point of which the *phase* of the disturbance is the same.

Let us consider the application of Huyghens' principle to calculate the progress of a given divergent wave. With any point P, at which the disturbance is required, as centre, describe a series of spheres of radii continually increasing by the constant difference $\frac{1}{2}\lambda$, the first of the series being of such radius (c) as to touch the given wave-surface at C. If R be the radius of curvature of the surface in any plane through P and C, the corresponding radius ρ of the outer boundary of the n^{th} zone is given by the equation

$$R + c = \sqrt{R^2 - \rho^2} + \sqrt{(c + \tfrac{1}{2}n\lambda)^2 - \rho^2},$$

from which we get approximately

$$\rho^2 = n\lambda \div \left(\frac{1}{R} + \frac{1}{c}\right) \text{......................} (1).$$

If the surface be one of revolution round PC, the area of the first n zones is $\pi\rho^2$, and since ρ^2 is proportional to n, it follows that the zones are of equal area. If the surface be not of revolution, the area of the first n zones is represented $\frac{1}{2}\int\rho^2 d\theta$, where θ is the azimuth of the plane in which ρ is measured, but it still remains true that the zones are of equal area. Since by hypothesis the normal motion does not vary rapidly over the wave-surface, the disturbances at P due to the various zones are nearly equal in magnitude and alternately opposite in sign, and we conclude that, as in the case of plane waves, the aggregate effect is the half of that due to the first zone. The phase at P is accordingly retarded behind that prevailing over the given wave-surface by an amount corresponding to the distance c.

The intensity of the disturbance at P depends upon the area of

the first Huyghens' zone, and upon the distance c. In the case of symmetry, we have

$$\frac{\pi\rho^2}{c}=\frac{\pi\lambda R}{R+c},$$

which shews that the disturbance is less than if R were infinite in the ratio $R+c : R$. This diminution is the effect of divergency, and is the same as would be obtained on the supposition that the motion is limited by a conical tube whose vertex is at the centre of curvature (§ 266). When the surface is not of revolution, the value of $\frac{1}{2}\int_0^{2\pi}\rho^2 d\theta \div c$ may be expressed in terms of the principal radii of curvature R_1 and R_2, with which R is connected by the relation

$$\frac{1}{R}=\frac{\cos^2\theta}{R_1}+\frac{\sin^2\theta}{R_2}.$$

We obtain on effecting the integration

$$\frac{1}{2c}\int_0^{2\pi}\rho^2 d\theta=\frac{\pi\lambda\sqrt{R_1R_2}}{\sqrt{(R_1+c)(R_2+c)}}\text{ .... (2)},$$

so that the amplitude is diminished by divergency in the ratio $\sqrt{(R_1+c)(R_2+c)} : \sqrt{R_1R_2}$, a result which might be anticipated by supposing the motion limited to a tube formed by normals drawn through a small contour traced on the wave-surface.

Although we have spoken hitherto of diverging waves only, the preceding expressions may also be applied to waves converging in one or in both of the principal planes, if we attach suitable signs to R_1 and R_2. In such a case the area of the first Huyghens' zone is greater than if the wave were plane, and the intensity of the vibration is correspondingly increased. If the point P coincide with one of the principal centres of curvature, the expression (2) becomes infinite. The investigation, on which (2) was founded, is then insufficient; all that we are entitled to affirm is that the disturbance is much greater at P than at other points on the same normal, that the disproportion increases with the frequency, and that it would become infinite for notes of infinitely high pitch, whose wave-length would be negligible in comparison with the distances concerned.

285. Huyghens' principle may also be applied to investigate the reflection of sound from curved surfaces. If the material surface of the reflector yielded so completely to the aërial

pressures that the normal motion at every point were the same as it would have been in the absence of the reflector, then the sound waves would pass on undisturbed. The reflection which actually ensues when the surface is unyielding may therefore be regarded as due to a normal motion of each element of the reflector, equal and opposite to that of the primary waves at the same point, and may be investigated by the formula proper to plane surfaces in the manner of the preceding section, and subject to a similar limitation as to the relative magnitudes of the wave-length and of the other distances concerned.

The most interesting case of reflection occurs when the surface is so shaped as to cause a concentration of rays upon a particular point (P). If the sound issue originally from a simple source at Q, and the surface be an ellipsoid of revolution having its foci at P and Q, the concentration is complete, the vibration reflected from every element of the surface being in the same phase on arrival at Q. If Q be infinitely distant, so that the incident waves are plane, the surface becomes a paraboloid having its focus at P, and its axis parallel to the incident rays. We must not suppose, however, that a symmetrical wave diverging from Q is converted by reflection at the ellipsoidal surface into a spherical wave converging symmetrically upon P; in fact, it is easy to see that the intensity of the convergent wave must be different in different directions. Nevertheless, when the wave-length is very small in comparison with the radius, the different parts of the convergent wave become approximately independent of one another, and their progress is not materially affected by the failure of perfect symmetry.

The increase of loudness due to curvature depends upon the area of reflecting surface, from which disturbances of uniform phase arrive, as compared with the area of the first Huyghens' zone of a plane reflector in the same position. If the distances of the reflector from the source and from the point of observation be considerable, and the wave-length be not very small, the first Huyghens' zone is already rather large, and therefore in the case of a reflector of moderate dimensions but little is gained by making it concave. On the other hand, in laboratory experiments, when the distances are moderate and the sounds employed are of high pitch, *e. g.* the ticking of a watch or the cracking of electric sparks, concave reflectors are very efficient and give a distinct concentration of sound on particular spots.

286. We have seen that if a ray proceeding from Q passes after reflection at a plane or curved surface through P, the point R at which it meets the surface is determined by the condition that $QR+RP$ is a minimum (or in some cases a maximum). The point R is then the centre of the system of Huyghens' zones; the amplitude of the vibration at P depends upon the area of the first zone, and its phase depends upon the distance $QR+RP$. If there be no point on the surface of the reflector, for which $QR+RP$ is a maximum or a minimum, the system of Huyghens' zones has no centre, and there is no ray proceeding from Q which arrives at P after reflection from the surface. In like manner if sound be reflected more than once, the course of a ray is determined by the condition that its whole length between any two points is a maximum or a minimum.

The same principle may be applied to investigate the *refraction* of sound in a medium, whose mechanical properties vary gradually from point to point. The variation is supposed to be so slow that no sensible reflection occurs, and this is not inconsistent with decided refraction of the rays in travelling distances which include a very great number of wave-lengths. It is evident that what we are now concerned with is not merely the length of the ray, but also the velocity with which the wave travels along it, inasmuch as this velocity is no longer constant. The condition to be satisfied is that the *time* occupied by a wave in travelling along a ray between any two points shall be a maximum or a minimum; so that, if V be the velocity of propagation at any point, and ds an element of the length of the ray, the condition may be expressed, $\delta\int V^{-1}dS=0$. This is Fermat's principle of least time.

The further developement of this part of the subject would lead us too far into the domain of geometrical optics. The fundamental assumption of the smallness of the wave-length, on which the doctrine of rays is built, having a far wider application to the phenomena of light than to those of sound, the task of developing its consequences may properly be left to the cultivators of the sister science. In the following sections the methods of optics are applied to one or two isolated questions, whose acoustical interest is sufficient to demand their consideration in the present work.

287. One of the most striking of the phenomena connected with the propagation of sound within closed buildings is that presented by "whispering galleries," of which a good and easily accessible example is to be found in the circular gallery at the base of the dome of St Paul's cathedral. As to the precise mode of action acoustical authorities are not entirely agreed. In the opinion of the Astronomer Royal[1] the effect is to be ascribed to reflection from the surface of the dome overhead, and is to be observed at the point of the gallery diametrically opposite to the source of sound. Every ray proceeding from a radiant point and reflected from the surface of a spherical reflector, will after reflection intersect that diameter of the sphere which contains the radiant point. This diameter is in fact a degraded form of one of the two caustic surfaces touched by systems of rays in general, being the loci of the centres of principal curvature of the surface to which the rays are normal. The concentration of rays on one diameter thus effected, does not require the proximity of the radiant point to the reflecting surface.

Judging from some observations that I have made in St Paul's whispering gallery, I am disposed to think that the principal phenomenon is to be explained somewhat differently. The abnormal loudness with which a whisper is heard is not confined to the position diametrically opposite to that occupied by the whisperer, and therefore, it would appear, does not depend materially upon the symmetry of the dome. The whisper seems to creep round the gallery horizontally, not necessarily along the shorter arc, but rather along that arc towards which the whisperer faces. This is a consequence of the very unequal audibility of a whisper in front of and behind the speaker, a phenomenon which may easily be observed in the open air[2].

Let us consider the course of the rays diverging from a radiant point P, situated near the surface of a reflecting sphere, and let us denote the centre of the sphere by O, and the diameter passing through P by AA', so that A is the point on the surface nearest to P. If we fix our attention on a ray which issues from P at an angle $\pm\theta$ with the tangent plane at A, we see that after any number of reflections it continues to touch a concentric sphere of radius $OP\cos\theta$, so that the whole conical pencil of rays which

[1] Airy *on Sound*, 2nd edition, 1871, p. 145.

[2] *Phil. Mag.* (5) III. p. 458, 1877.

originally make angles with the tangent plane at A numerically less than θ, is ever afterwards included between the reflecting surface and that of the concentric sphere of radius $OP \cos\theta$. The usual divergence in three dimensions entailing a diminishing intensity varying as r^{-2} is replaced by a divergence in two dimensions, like that of waves issuing from a source situated between two parallel reflecting planes, with an intensity varying as r^{-1}. The less rapid enfeeblement of sound by distance than that usually experienced is the leading feature in the phenomena of whispering galleries.

The thickness of the sheet included between the two spheres becomes less and less as A approaches P, and in the limiting case of a radiant point situated on the surface of the reflector is expressed by $OA(1-\cos\theta)$, or, if θ be small, $\frac{1}{2}OA \,.\, \theta^2$ approximately. The solid angle of the pencil, which determines the whole amount of radiation in the sheet, is $4\pi\theta$; so that as θ is diminished without limit the intensity becomes infinite, as compared with the intensity at a finite distance from a similar source in the open.

It is evident that this clinging, so to speak, of sound to the surface of a concave wall does not depend upon the exactness of the spherical form. But in the case of a true sphere, or rather of any surface symmetrical with respect to AA', there is in addition the other kind of concentration spoken of at the commencement of the present section which is peculiar to the point A' diametrically opposite to the source. It is probable that in the case of a nearly spherical dome like that of St Paul's a part of the observed effect depends upon the symmetry, though perhaps the greater part is referable simply to the general concavity of the walls.

The propagation of earthquake disturbances is probably affected by the curvature of the surface of the globe acting like a whispering gallery, and perhaps even sonorous vibrations generated at the surface of the land or water do not entirely escape the same kind of influence.

In connection with the acoustics of public buildings there are many points which still remain obscure. It is important to bear in mind that the loss of sound in a single reflection at a smooth wall is very small, whether the wall be plane or curved. In order to prevent reverberation it may often be necessary to introduce

carpets or hangings to absorb the sound. In some cases the presence of an audience is found sufficient to produce the desired effect. In the absence of all deadening material the prolongation of sound may be very considerable, of which perhaps the most striking example is that afforded by the Baptistery at Pisa, where the notes of the common chord sung consecutively may be heard ringing on together for many seconds. According to Henry[1] it is important to prevent the repeated reflection of sound backwards and forwards along the *length* of a hall intended for public speaking, which may be accomplished by suitably placed oblique surfaces. In this way the number of reflections in a given time is increased, and the undue prolongation of sound is checked.

288. Almost the only instance of acoustical refraction, which has a practical interest, is the deviation of sonorous rays from a rectilinear course due to heterogeneity of the atmosphere. The variation of pressure at different levels does not of itself give rise to refraction, since the velocity of sound is independent of density; but, as was first pointed out by Prof. Osborne Reynolds[2], the case is different with the variations of temperature which are usually to be met with. The temperature of the atmosphere is determined principally by the condensation or rarefaction, which any portion of air must undergo in its passage from one level to another, and its normal state is one of "convective equilibrium[3]," rather than of uniformity. According to this view the relation between pressure and density is that expressed in (9) § 246, and the velocity of sound is given by

$$V^2 = \frac{dp}{d\rho} = \gamma \frac{p_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \text{..................... (1).}$$

To connect the pressure and density with the elevation (z), we have the hydrostatical equation

$$dp = -g\rho\, dz \text{ .... (2),}$$

from which and (1) we find

$$V^2 = V_0^2 - (\gamma - 1)\, gz \text{ .... (3),}$$

if V_0 be the velocity at the surface. The corresponding relation

[1] *Amer. Assoc. Proc.* 1856, p. 119.

[2] *Proceedings of the Royal Society*, Vol. XXII. p. 531. 1874.

[3] Thomson, *On the convective equilibrium of temperature in the atmosphere. Manchester Memoirs*, 1861—62.

between temperature and elevation obtained by means of equation (10) § 246 is

$$\frac{\theta}{\theta_0} = 1 - \frac{\gamma - 1}{V_0^2} gz \qquad (4),$$

where θ_0 is the temperature at the surface.

According to (4) the fall of temperature would be about 1° Cent. in 330 feet, which does not differ much from the results of Glaisher's balloon observations. When the sky is clear, the fall of temperature during the day is more rapid than when the sky is cloudy, but towards sunset the temperature becomes approximately constant[1]. Probably on clear nights it is often warmer above than below.

The explanation of acoustical refraction as dependent upon a variation of temperature with height is almost exactly the same as that of the optical phenomenon of mirage. The curvature (ρ^{-1}) of a ray, whose course is approximately horizontal, is easily estimated by the method given by Prof. James Thomson[2]. Normal planes drawn at two consecutive points along the ray meet at the centre of curvature and are tangential to the wave-surface in its two consecutive positions. The portions of rays at elevations z and $z + \delta z$ respectively intercepted between the normal planes are to one another in the ratio $\rho : \rho - \delta z$, and also, since they are described in the same time, in the ratio $V : V + \delta V$. Hence in the limit

$$\frac{1}{\rho} = -\frac{d \log V}{dz} \qquad (5).$$

In the normal state of the atmosphere a ray, which starts horizontally, turns gradually upwards, and at a sufficient distance passes over the head of an observer whose station is at the same level as the source. If the source be elevated, the sound is heard at the surface of the earth by means of a ray which starts with a downward inclination; but, if both the observer and the source be on the surface, there is no direct ray, and the sound is heard, if at all, by means of diffraction. The observer may then be said to be situated in a sound shadow, although there may be no obstacle in the direct line between himself and the source. According to (3)

[1] *Nature*, Sept. 20, 1877.

[2] See Everett, *On the Optics of Mirage*. *Phil. Mag.* (4) XLV. pp. 161, 248.

$$2V\frac{dV}{dz} = -(\gamma - 1)g,$$

so that $$\rho = \frac{2V^2}{(\gamma-1)g} = \frac{4}{\gamma-1}\cdot\frac{V^2}{2g} \quad\text{..................}(6);$$

or the radius of curvature of a horizontal ray is about ten times the height through which a body must fall under the action of gravity in order to acquire a velocity equal to the velocity of sound. If the elevations of the observer and of the source be z_1 and z_2, the greatest distance at which the sound can be heard otherwise than by diffraction is

$$\sqrt{(2z_1\rho)} + \sqrt{(2z_2\rho)} \quad\text{......................}(7).$$

It is not to be supposed that the condition of the atmosphere is always such that the relation between velocity and elevation is that expressed in (3). When the sun is shining, the variation of temperature upwards is more rapid; on the other hand, as Prof. Reynolds has remarked, when rain is falling, a much slower variation is to be expected. In the arctic regions, where the nights are long and still, radiation may have more influence than convection in determining the equilibrium of temperature, and if so the propagation of sound in a horizontal direction would be favoured by the approximately isothermal condition of the atmosphere.

The general differential equation for the path of a ray, when the surfaces of equal velocity are parallel planes, is readily obtained from the law of sines. If θ be the angle of incidence, $V \div \sin\theta$ is not altered by a refracting surface, and therefore in the case supposed remains constant along the whole course of a ray. If x be the horizontal co-ordinate, and the constant value of $V \div \sin\theta$ be called c, we get

$$\frac{dx}{dz} = \frac{V}{\sqrt{c^2 - V^2}},$$

or $$x = \int \frac{V\,dz}{\sqrt{c^2 - V^2}} \quad\text{......................}(8).$$

If the law of velocity be that expressed in (3),

$$dz = -\frac{2V\,dV}{(\gamma-1)g},$$

and thus $$x = -\frac{2}{(\gamma-1)g}\int\frac{V^2\,dV}{\sqrt{c^2 - V^2}},$$

or, on effecting the integration,

$$(\gamma-1)\,g\,x = \text{constant} + V\sqrt{c^2 - V^2} - c^2 \sin^{-1}\frac{V}{c} \;\ldots\ldots(9),$$

in which V may be expressed in terms of z by (3).

A simpler result will be obtained by taking an approximate form of (3), which will be accurate enough to represent the cases of practical interest. Neglecting the square and higher powers of z, we may take

$$V^{-1} = V_0^{-1} + \frac{g(\gamma-1)z}{2V_0^3} \;\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Writing for brevity β in place of $\dfrac{g(\gamma-1)}{2V_0^3}$, we have $\beta\,dz = dV^{-1}$. By substitution in (8)

$$c\beta x = \int \frac{d\left(\frac{c}{V}\right)}{\sqrt{\frac{c^2}{V^2}-1}} = \log\left[\frac{c}{V} + \sqrt{\frac{c^2}{V^2}-1}\right] \ldots\ldots(11),$$

the origin of x being taken so as to correspond with $V=c$, that is at the place where the ray is horizontal. Expressing V in terms of x, we find

$$\frac{2c}{V} = e^{c\beta x} + e^{-c\beta x},$$

whence $$\beta z = -V_0^{-1} + \frac{1}{2c}\left(e^{c\beta x} + e^{-c\beta x}\right) \ldots\ldots\ldots\ldots\ldots(12).$$

The path of each ray is therefore a catenary whose vertex is downwards; the linear parameter is $\dfrac{2V_0^3}{g(\gamma-1)c}$, and varies from ray to ray.

289. Another cause of atmospheric refraction is to be found in the action of wind. It has long been known that sounds are generally better heard to leeward than to windward of the source; but the fact remained unexplained until Stokes[1] pointed out that the increasing velocity of the wind overhead must interfere with the rectilinear propagation of sound rays. From Fermat's law of least time it follows that the course of a ray in a moving, but

[1] *Brit. Ass. Rep.* 1857, p. 22.

otherwise homogeneous, medium, is the same as it would be in a medium, of which all the parts are at rest, if the velocity of propagation be increased at every point by the component of the wind-velocity in the direction of the ray. If the wind be horizontal, and do not vary in the same horizontal plane, the course of a ray, whose direction is everywhere but slightly inclined to that of the wind, may be calculated on the same principles as were applied in the preceding section to the case of a variable temperature, the normal velocity of propagation at any point being increased, or diminished, by the local wind-velocity, according as the motion of the sound is to leeward or to windward. Thus, when the wind increases overhead, which may be looked upon as the normal state of things, a horizontal ray travelling to windward is gradually bent upwards, and at a moderate distance passes over the head of an observer; rays travelling with the wind, on the other hand, are bent downwards, so that an observer to leeward of the source hears by a direct ray which starts with a slight upward inclination, and has the advantage of being out of the way of obstructions for the greater part of its course.

The law of refraction at a horizontal surface, in crossing which the velocity of the wind changes discontinuously, is easily investigated. It will be sufficient to consider the case in which the direction of the wind and the ray are in the same vertical plane. If θ be the angle of incidence, which is also the angle between the plane of the wave and the surface of separation, U be the velocity of the air in that direction which makes the smaller angle with the ray, and V be the common velocity of propagation, the velocity of the trace of the plane of the wave on the surface of separation is

$$\frac{V}{\sin\theta} + U \text{ .... (1),}$$

which quantity is unchanged by the refraction. If therefore U' be the velocity of the wind on the second side, and θ' be the angle of refraction,

$$\frac{V}{\sin\theta} + U = \frac{V}{\sin\theta'} + U' \text{ .... (2),}$$

which differs from the ordinary optical law. If the wind-velocity vary continuously, the course of a ray may be calculated from the condition that the expression (1) remains constant.

If we suppose that $U=0$, the greatest admissible value of U' is

$$U' = V\{\operatorname{cosec}\theta - 1\} \quad \text{(3)}.$$

At a stratum where U' has this value, the direction of the ray which started at an angle θ has become parallel to the refracting surfaces, and a stratum where U' has a greater value cannot be penetrated at all. Thus a ray travelling upwards in still air at an inclination $(\frac{1}{2}\pi - \theta)$ to the horizon is reflected by a wind overhead of velocity exceeding that given in (3), and this independently of the velocities of intermediate strata. To take a numerical example, all rays whose upward inclination is less than 11°, are totally reflected by a wind of the same azimuth moving at the moderate speed of 15 miles per hour. The effects of such a wind on the propagation of sound cannot fail to be very important. Over the surface of still water sound moving to leeward, being confined between parallel reflecting planes, diverges in two dimensions only, and may therefore be heard at distances far greater than would otherwise be possible. Another possible effect of the reflector overhead is to render sounds audible which in still air would be intercepted by hills or other obstacles intervening. For the production of these phenomena it is not necessary that there be absence of wind at the source of sound, but, as appears at once from the form of (2), merely that the *difference* of velocities $U' - U$ attain a sufficient value.

The differential equation to the path of a ray, when the wind-velocity U is continuously variable, is

$$V\sqrt{1 + \left(\frac{dz}{dx}\right)^2} = c \pm U \quad \text{(4)},$$

whence

$$x = \int \frac{V\,dz}{\sqrt{(c \pm U)^2 - V^2}} \quad \text{(5)}.$$

In comparing (5) with (8) of the preceding section, which is the corresponding equation for ordinary refraction, we must remember that V is now constant. If, for the sake of obtaining a definite result, we suppose that the law of variation of wind at different levels is that expressed by

$$U = \alpha + \beta z \quad \text{(6)},$$

we have

$$\beta x = V\int \frac{dU}{\sqrt{(c \pm U)^2 - V^2}} \quad \text{(7)},$$

which is of the same form as (11) of the preceding section. The course of a ray is accordingly a catenary in the present case also, but there is a most important distinction between the two problems. When the refraction is of the ordinary kind, depending upon a variable velocity of propagation, the direction of a ray may be reversed. In the case of atmospheric refraction, due to a diminution of temperature upwards, the course of a ray is a catenary, whose vertex is downwards, in whichever direction the ray may be propagated. When the refraction is due to wind, whose velocity increases upwards, according to the law expressed in (6) with β positive, the path of a ray, whose direction is upward, is also along a catenary with vertex downwards, but a ray whose direction is downward cannot travel along this path. In the latter case the vertex of the catenary along which the ray travels is directed upwards.

290. In the paper by Reynolds already referred to, an account is given of some interesting experiments especially directed to test the theory of refraction by wind. It was found that "In the direction of the wind, when it was strong, the sound (of an electric bell) could be heard as well with the head on the ground as when raised, even when in a hollow with the bell hidden from view by the slope of the ground; and no advantage whatever was gained either by ascending to an elevation or raising the bell. Thus, with the wind over the grass the sound could be heard 140 yards, and over snow 360 yards, either with the head lifted or on the ground; whereas at right angles to the wind on all occasions the range was extended by raising either the observer or the bell."

"Elevation was found to affect the range of sound against the wind in a much more marked manner than at right angles."

"Over the grass no sound could be heard with the head on the ground at 20 yards from the bell, and at 30 yards it was lost with the head 3 feet from the ground, and its full intensity was lost when standing erect at 30 yards. At 70 yards, when standing erect, the sound was lost at long intervals, and was only faintly heard even then; but it became continuous again when the ear was raised 9 feet from the ground, and it reached its full intensity at an elevation of 12 feet."

Prof. Reynolds thus sums up the results of his experiments:—

1. "When there is no wind, sound proceeding over a rough surface is more intense above than below."

2. "As long as the velocity of the wind is greater above than below, sound is lifted up to windward and is not destroyed."

3. "Under the same circumstances it is brought down to leeward, and hence its range extended at the surface of the ground."

Atmospheric refraction has an important bearing on the audibility of fog-signals, a subject which within the last few years has occupied the attention of two eminent physicists, Prof. Henry in America and Prof. Tyndall in this country. Henry[1] attributes almost all the vagaries of distant sounds to refraction, and has shewn how it is possible by various suppositions as to the motion of the air overhead to explain certain abnormal phenomena which have come under the notice of himself and other observers, while Tyndall[2], whose investigations have been equally extensive, considers the very limited distances to which sounds are sometimes audible to be due to an actual stopping of the sound by a flocculent condition of the atmosphere arising from unequal heating or moisture. That the latter cause is capable of operating in this direction to a certain extent cannot be doubted. Tyndall has proved by laboratory experiments that the sound of an electric bell may be sensibly intercepted by alternate layers of gases of different densities; and, although it must be admitted that the alternations of density were both more considerable and more abrupt than can well be supposed to occur in the open air, except perhaps in the immediate neighbourhood of the solid ground, some of the observations on fog-signals themselves seem to point directly to the explanation in question.

Thus it was found that the blast of a siren placed on the summit of a cliff overlooking the sea was followed by an echo of gradually diminishing intensity, whose duration sometimes amounted to as much as 15 seconds. This phenomenon was observed "when the sea was of glassy smoothness," and cannot apparently be attributed to any other cause than that assigned to it by Tyndall. It is therefore probable that refraction and acoustical opacity are both concerned in the capricious behaviour of fog-signals. *A priori* we should certainly be disposed to attach the greater importance to refraction, and Reynolds has shewn that some of Tyndall's own observations admit of explanation upon this

[1] Report of the Lighthouse Board of the United States for the year 1874.
[2] *Phil. Trans.* 1874. *Sound*, 3rd edition, Ch. VII.

principle. A failure in *reciprocity* can only be explained in accordance with theory by the action of wind (§ 111).

According to the hypothesis of acoustic clouds, a difference might be expected in the behaviour of sounds of long and of short duration, which it may be worth while to point out here, as it does not appear to have been noticed by any previous writer. Since energy is not lost in reflection and refraction, the intensity of radiation at a given distance from a continuous source of sound (or light) is not altered by an enveloping cloud of spherical form and of uniform density, the loss due to the intervening parts of the cloud being compensated by reflection from those which lie beyond the source. When, however, the sound is of short duration, the intensity at a distance may be very much diminished by the cloud on account of the different distances of its reflecting parts and the consequent drawing out of the sound, although the whole intensity, as measured by the time-integral, may be the same as if there had been no cloud at all. This is perhaps the explanation of Tyndall's observation, that different kinds of signals do not always preserve the same order of effectiveness. In some states of the weather a "howitzer firing a 3-lb. charge commanded a larger range than the whistles, trumpets, or syren," while on other days "the inferiority of the gun to the syren was demonstrated in the clearest manner." It should be noticed, however, that in the same series of experiments it was found that the liability of the sound of a gun "to be quenched or deflected by an opposing wind, so as to be practically useless at a very short distance to windward, is very remarkable." The refraction proper must be the same for all kinds of sounds, but for the reason explained above, the diffraction round the edge of an obstacle may be less effective for the report of a gun than for the sustained note of a siren.

Another point examined by Tyndall was the influence of fog on the propagation of sound. In spite of isolated assertions to the contrary[1], it was generally believed on the authority of Derham that the influence of fog was prejudicial. Tyndall's observations prove satisfactorily that this opinion is erroneous, and that the passage of sound is favoured by the homogeneous condition of the atmosphere which is the usual concomitant of foggy weather. When the air is saturated with moisture, the fall of temperature with elevation according to the law of convective equilibrium is

[1] See for example Desor, *Fortschritte der Physik*, XI. p. 217. 1855.

much less rapid than in the case of dry air, on account of the *condensation of vapour which then accompanies expansion. From a calculation by Thomson*[1] *it* appears *that* in warm *fog the* effect of evaporation and condensation would be to diminish the fall of temperature by one-half. The acoustical refraction due to temperature would thus be lessened, and in other respects no doubt the condition of the air would be favourable to the propagation of sound, provided no obstruction were offered by the suspended particles themselves. In a future chapter we shall investigate the disturbance of plane sonorous waves by a small obstacle, and we shall find that the effect depends upon the ratio of the diameter of the obstacle to the wave-length of the sound.

The reader who is desirous of pursuing this subject may consult a paper by Reynolds "On the Refraction of Sound by the Atmosphere[2]," as well as the authorities already referred to. It may be mentioned that Reynolds agrees with Henry in considering refraction to be the really important cause of disturbance, but further observations are much needed. See also § 294.

291. On the assumption that the disturbance at an aperture in a screen is the same as it would have been at the same place in the absence of the screen, we may solve various problems respecting the diffraction of sound by the same methods as are employed for the corresponding problems in physical optics. For example, the disturbance at a distance on the further side of an infinite plane wall, pierced with a circular aperture on which plane waves of sound impinge directly, may be calculated as in the analogous problem of the diffraction pattern formed at the focus of a circular object-glass. Thus in the case of a symmetrical speaking trumpet the sound is a maximum along the axis of the instrument, where all the elementary disturbances issuing from the various points of the plane of the mouth are in one phase. In oblique directions the intensity is less; but it does not fall materially short of the maximum value until the obliquity is such that the difference of distances of the nearest and furthest points of the mouth amounts to about half a wave-length. At a somewhat greater obliquity the mouth may be divided into two parts, of which the nearer gives an aggregate effect equal in magnitude,

[1] *Manchester Memoirs*, 1861—62.

[2] *Phil. Trans.* Vol. 166, p. 315. 1876.

but opposite in phase, to that of the further; so that the intensity in this direction vanishes. In directions still more oblique the sound revives, increases to an intensity equal to ·017 of that along the axis[1], again diminishes to zero, and so on, the alternations corresponding to the bright and dark rings which surround the central patch of light in the image of a star. If R denote the radius of the mouth, the angle, at which the first silence occurs, is $\sin^{-1}\left(\cdot 610\,\frac{\lambda}{R}\right)$. When the diameter of the mouth does not exceed $\frac{1}{2}\lambda$, the elementary disturbances combine without any considerable antagonism of phase, and the intensity is nearly uniform in all directions. It appears that concentration of sound along the axis requires that the ratio $R:\lambda$ should be large, a condition not usually satisfied in the ordinary use of speaking trumpets, whose efficiency depends rather upon an increase in the original volume of sound (§ 280). When, however, the vibrations are of very short wave-length, a trumpet of moderate size is capable of effecting a considerable concentration along the axis, as I have myself verified in the case of a hiss.

292. Although such calculations as those referred to in the preceding section are useful as giving us a general idea of the phenomena of diffraction, it must not be forgotten that the auxiliary assumption on which they are founded is by no means strictly and generally true. Thus in the case of a wave directly incident upon a screen the normal velocity in the plane of the aperture is not constant, as has been supposed, but increases from the centre towards the edge, becoming infinite at the edge itself. In order to investigate the conditions by which the actual velocity is determined, let us for the moment suppose that the aperture is filled up. The incident wave $\phi = \cos(nt - \kappa x)$ is then perfectly reflected, and the velocity-potential on the negative side of the screen $(x = 0)$ is

$$\phi = \cos(nt - \kappa x) + \cos(nt + \kappa x) \quad\ldots\ldots\ldots\ldots\ldots (1),$$

giving, when $x = 0$, $\phi = 2\cos nt$. This corresponds to the vanishing of the normal velocity over the area of the aperture; the completion of the problem requires us to determine a variable normal velocity over the aperture such that the potential due to it (§ 278) shall increase by the constant quantity $2\cos nt$ in crossing

[1] Verdet, *Leçons d'optique physique*, t. I p. 306.

from the negative to the positive side; or, since the crossing involves simply a change of sign, to determine a value of the normal velocity over the area of the aperture which shall give on the positive side $\phi = \cos nt$ over the same area. The result of superposing the two motions thus defined satisfies all the conditions of the problem, giving the same velocity and pressure on the two sides of the aperture, and a vanishing normal velocity over the remainder of the screen.

If $P \cos(nt + \epsilon)$ denote the value of $\frac{d\phi}{dx}$ at the various points of the area (S) of the aperture, the condition for determining P and ϵ is by (6) § 278,

$$-\frac{1}{2\pi}\iint P \frac{\cos(nt - \kappa r + \epsilon)}{r}\, dS = \cos nt \quad \text{........(2)},$$

where r denotes the distance between the element dS and any fixed point in the aperture. When P and ϵ are known, the complete value of ϕ for any point on the positive side of the screen is given by

$$\phi = -\frac{1}{2\pi}\iint P \frac{\cos(nt - \kappa r + \epsilon)}{r}\, dS \quad \text{........................(3)},$$

and for any point on the negative side by

$$\phi = +\frac{1}{2\pi}\iint P \frac{\cos(nt - \kappa r + \epsilon)}{r}\, dS + 2\cos nt \cos \kappa x \quad \text{......(4)}.$$

The expression of P and ϵ for a finite aperture, even if of circular form, is probably beyond the power of known methods; but in the case where the dimensions are very small in comparison with the wave-length the solution of the problem may be effected for the circle and the ellipse. If r be the distance between two points, both of which are situated in the aperture, κr may be neglected, and we then obtain from (2)

$$\epsilon = 0, \qquad 1 = -\frac{1}{2\pi}\iint P \frac{dS}{r} \quad \text{.....................(5)},$$

shewing that $-\frac{P}{2\pi}$ is the density of the matter which must be distributed over S in order to produce there the constant potential unity. At a distance from the opening on the positive side we may consider r as constant, and take

$$\phi = M \frac{\cos(nt - \kappa r)}{r} \quad \text{.....................(6)},$$

where $M = -\frac{1}{2\pi}\iint P\,dS$, denoting the total quantity of matter which must be supposed to be distributed. It will be shewn on a future page that for an ellipse of semimajor axis a, and eccentricity e,

$$M = a \div F(e) \qquad (7),$$

where F is the symbol of the complete elliptic function of the first kind. In the case of a circle, $F(e) = \frac{1}{2}\pi$, and

$$M = \frac{2a}{\pi} \qquad (8).$$

This result is quite different from that which we should obtain on the hypothesis that the normal velocity in the aperture has the value proper to the primary wave. In that case by (3) § 283

$$\phi = -\frac{\pi a^2}{\lambda}\frac{\sin(nt - \kappa r)}{r} \qquad (9).$$

If there be several small apertures, whose distances apart are much greater than their dimensions, the same method gives

$$\phi = M_1\frac{\cos(nt - \kappa r_1)}{r_1} + M_2\frac{\cos(nt - \kappa r_2)}{r_2} + \ldots\ldots \qquad (10).$$

The diffraction of sound is a subject which has attracted but little attention either from mathematicians or experimentalists. Although the general character of the phenomena is well understood, and therefore no very startling discoveries are to be expected, the exact theoretical solution of a few of the simpler problems, which the subject presents, would be interesting; and, even with the present imperfect methods, something probably might be done in the way of experimental examination.

293. The value of a function ϕ which satisfies $\nabla^2\phi = 0$ throughout the interior of a simply-connected closed space S can be expressed as the potential of matter distributed over the surface of S. In a certain sense this is also true of the class of functions with which we are now occupied, which satisfy $\nabla^2\phi + \kappa^2\phi = 0$. The following is Helmholtz's proof[1]. By Green's theorem, if ϕ and ψ denote any two functions of x, y, z,

$$\iint \phi\frac{d\psi}{dn}\,dS - \iiint \phi\,\nabla^2\psi\,dV = \iint \psi\frac{d\phi}{dn}\,dS - \iiint \psi\,\nabla^2\phi\,dV.$$

[1] *Theorie der Luftschwingungen in Röhren mit offenen Enden.* Crelle, Bd. LVII. p. 1. 1860.

To each side add $-\iiint \kappa^2 \phi\psi \, dV$; then if

$$a^2(\nabla^2\phi + \kappa^2\phi) + \Phi = 0, \qquad a^2(\nabla^2\psi + \kappa^2\psi) + \Psi = 0,$$

$$a^2 \iint \phi \frac{d\psi}{dn} dS + \iiint \phi \Psi \, dV = a^2 \iint \psi \frac{d\phi}{dn} dS + \iiint \psi \Phi \, dV \quad \text{...(1).}$$

If Φ and Ψ vanish within S, we have simply

$$\iint \phi \frac{d\psi}{dn} dS = \iint \psi \frac{d\phi}{dn} dS \quad \text{......................(2).}$$

Suppose, however, that

$$\phi = \frac{e^{-i\kappa r}}{r} \quad \text{..............................(3),}$$

where r represents the distance of any point from a fixed origin O within S. At all points, except O, Φ vanishes; and the last term in (1) becomes

$$\iiint \psi \Phi \, dV = -a^2 \iiint \psi \nabla^2 \left(\frac{1}{r}\right) dV = 4\pi a^2 \psi,$$

ψ referring to the point O. Thus

$$4\pi\psi = \iint \frac{d\psi}{dn} \frac{e^{-i\kappa r}}{r} dS - \iint \psi \frac{d}{dn}\left(\frac{e^{-i\kappa r}}{r}\right) dS$$
$$+ \frac{1}{a^2} \iiint \Psi \frac{e^{-i\kappa r}}{r} dV \quad \text{...........(4),}$$

in which if Ψ vanish, we have an expression for the value of ψ at any interior point O in terms of the surface values of ψ and of $\frac{d\psi}{dn}$. In the case of the common potential, on which we fall back by putting $\kappa = 0$, ψ would be determined by the surface values of $\frac{d\psi}{dn}$ only. But with κ finite, this law ceases to be universally true. For a given space S there is, as in the case investigated in § 267, a series of determinate values of κ, corresponding to the periods of the possible modes of simple harmonic vibration, which may take place within a closed rigid envelope having the form of S. With any of these values of κ, it is obvious that ψ cannot be determined by its normal variation over S, and the fact that it satisfies throughout S the equation $\nabla^2\psi + \kappa^2\psi = 0$. But if the supposed value of κ do not coincide with one of the series, then the problem

is determinate; for the difference of any two possible solutions, if finite, would satisfy the condition of giving no normal velocity over S, a condition which by hypothesis cannot be satisfied with the assumed value of κ.

If the dimensions of the space S be very small in comparison with $\lambda\,(=2\pi\div\kappa)$, $e^{-i\kappa r}$ may be replaced by unity; and we learn that ψ differs but little from a function which satisfies throughout S the equation $\nabla^2\phi=0$.

294. On his extension of Green's theorem (1) Helmholtz founds his proof of the important theorem contained in the following statement: *If in a space filled with air which is partly bounded by finitely extended fixed bodies and is partly unbounded, sound waves be excited at any point* A, *the resulting velocity-potential at a second point* B *is the same both in magnitude and phase, as it would have been at* A, *had* B *been the source of the sound.*

If the equation

$$a^2\iint\left(\phi\frac{d\psi}{dn}-\psi\frac{d\phi}{dn}\right)dS=\iiint(\psi\Phi-\phi\Psi)\,dV\,\ldots\ldots(1),$$

in which ϕ and ψ are arbitrary functions, and

$$\Phi=-a^2(\nabla^2\phi+\kappa^2\phi),\qquad \Psi=-a^2(\nabla^2\psi+\kappa^2\psi),$$

be applied to a space completely enclosed by a rigid boundary and containing any number of detached rigid fixed bodies, and if ϕ, ψ be velocity-potentials due to sources within S, we get

$$\iiint(\psi\Phi-\phi\Psi)\,dV=0\,\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Thus, if ϕ be due to a source concentrated in one point A, $\Phi=0$ except at that point, and

$$\iiint\psi\Phi\,dV=\psi_A\iiint\Phi\,dV,$$

where $\iiint\Phi\,dV$ represents the intensity of the source. Similarly, if ψ be due to a source situated at B,

$$\iiint\phi\Psi\,dV=\phi_B\iiint\Psi\,dV.$$

Accordingly, if the sources be finite and equal, so that

$$\iiint\Phi\,dV=\iiint\Psi\,dV\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

it follows that

$$\psi_A = \phi_B \text{.....................................} (4),$$

which is the symbolical statement of Helmholtz's theorem.

If the space S extend to infinity, the surface integral still vanishes, and the result is the same; but it is not necessary to go into detail here, as this theorem is included in the vastly more general principle of reciprocity established in Chapter V. The investigation there given shews that the principle remains true in the presence of dissipative forces, provided that these arise from resistances varying as the first power of the velocity, that the fluid need not be homogeneous, nor the neighbouring bodies rigid or fixed. In the application to infinite space, all obscurity is avoided by supposing the vibrations to be slowly dissipated after having escaped to a distance from A and B, the sources under contemplation.

The reader must carefully remember that in this theorem equal sources of sound are those produced by the periodic introduction and abstraction of equal quantities of fluid, or something whose effect is the same, and that equal sources do not necessarily evolve equal amounts of energy in equal times. For instance, a source close to the surface of a large obstacle emits twice as much energy as an equal source situated in the open.

As an example of the use of this theorem we may take the case of a hearing, or speaking, trumpet consisting of a conical tube, whose efficiency is thus seen to be the same, whether a sound produced at a point outside is observed at the vertex of the cone, or a source of equal strength situated at the vertex is observed at the external point.

It is important also to bear in mind that Helmholtz's form of the reciprocity theorem is applicable only to *simple* sources of sound, which in the absence of obstacles would generate symmetrical waves. As we shall see more clearly in a subsequent chapter, it is possible to have sources of sound, which, though concentrated in an infinitely small region, do not satisfy this condition. It will be sufficient here to consider the case of *double* sources, for which the modified reciprocal theorem has an interest of its own.

Let us suppose that A is a simple source, giving at a point B the potential $-\psi$, and that A' is an equal and opposite source situated at a neighbouring point, whose potential at B is $\psi + \Delta\psi$.

If both sources be in operation simultaneously, the potential at B is $\Delta\psi$. Now let us suppose that there is a simple source at B, whose intensity and phase are the same as those of the sources at A and A'; the resulting potential at A is ψ, and at A' $\psi+\Delta\psi$. If the distance AA' be denoted by h, and be supposed to diminish without limit, the velocity of the fluid at A in the direction AA' is the limit of $\Delta\psi : h$. Hence, if we define a unit double source as the limit of two equal and opposite simple sources whose distance is diminished, and whose intensity is increased without limit in such a manner that the product of the intensity and the distance is the same as for two unit simple sources placed at the unit distance apart, we may say that the velocity of the fluid at A in direction AA' due to a unit simple source at B is numerically equal to the potential at B due to a unit *double* source at A, whose axis is in the direction AA'. This theorem, be it observed, is true in spite of any obstacles or reflectors that may exist in the neighbourhood of the sources.

Again, if AA' and BB' represent two unit double sources of the same phase, the velocity at B in direction BB' due to the source AA' is the same as the velocity at A in direction AA' due to the source BB'. These and other results of a like character may also be obtained on an immediate application of the general principle of § 108. These examples will be sufficient to shew that in applying the principle of reciprocity it is necessary to attend to the character of the sources. A double source, situated in an open space, is inaudible from any point in its equatorial plane, but it does not follow that a simple source in the equatorial plane is inaudible from the position of the double source. On this principle, I believe, may be explained a curious experiment by Tyndall[1], in which there was an apparent failure of reciprocity[2]. The source of sound employed was a reed of very high pitch, mounted in a tube, along whose axis the intensity was considerably greater than in oblique directions.

295. The kinetic energy T of the motion within a closed surface S is expressed by

$$T=\tfrac{1}{2}\rho_0\iiint\Sigma\left(\frac{d\phi}{dx}\right)^2 dV \quad\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

[1] *Proceedings of the Royal Institution*, Jan. 1875. Also Tyndall, *On Sound*, 3rd edition, p. 405.

[2] See a note "On the Application of the Principle of Reciprocity to Acoustics." *Royal Society Proceedings*, Vol. XXV. p. 118, 1876, or *Phil. Mag.* (5) III. p. 300.

so that
$$\frac{dT}{dt} = \rho_0 \iiint \Sigma \frac{d\phi}{dx}\frac{d\dot{\phi}}{dx}\, dV$$
$$= \rho_0 \iint \dot{\phi} \frac{d\phi}{dn} dS - \rho_0 \iiint \dot{\phi}\, \nabla^2\phi\, dV \qquad\qquad (2),$$

by Green's theorem. For the potential energy V_1 we have by (12) § 245

$$V_1 = \frac{\rho_0}{2a^2}\iiint \dot{\phi}^2 dV \qquad\qquad (3),$$

whence
$$\frac{dV_1}{dt} = \frac{\rho_0}{a^2}\iiint \dot{\phi}\ddot{\phi}\, dV = \frac{\rho_0}{a^2}\iiint \left\{\frac{dR}{dt} + a^2\nabla^2\phi\right\}\dot{\phi}\, dV \ldots (4),$$

by the general equation of motion (9) § 244. Thus, if E denote the whole energy within the space S,

$$\frac{dE}{dt} = \rho_0 \iint \dot{\phi}\frac{d\phi}{dn} dS + \frac{\rho_0}{a^2}\iiint \frac{dR}{dt}\dot{\phi}\, dV \qquad\qquad (5),$$

of which the first term represents the work transmitted across the boundary S, and the second represents the work done by internal sources of sound.

If the boundary S be a fixed rigid envelope, and there be no internal sources, E retains its initial value throughout the motion. This principle has been applied by Kirchhoff[1] to prove the determinateness of the motion resulting from given arbitrary initial conditions. Since every element of E is positive, there can be no motion within S, if E be zero. Now, if there were two motions possible corresponding to the same initial conditions, their difference would be a motion for which the initial value of E was zero; but by what has just been said such a motion cannot exist.

[1] *Vorlesungen über Math. Physik*, p. 311.

CHAPTER XV.

FURTHER APPLICATION OF THE GENERAL EQUATIONS.

296. WHEN a train of plane waves, otherwise unimpeded, impinges upon a space occupied by matter, whose mechanical properties differ from those of the surrounding medium, secondary waves are thrown off, which may be regarded as a disturbance due to the change in the nature of the medium—a point of view more especially appropriate, when the *region of disturbance*, as well as the alteration of mechanical properties, is small. If the medium and the obstacle be fluid, the mechanical properties spoken of are two—the *compressibility* and the *density*: no account is here taken of friction or viscosity. In the chapter on spherical harmonic analysis we shall consider the problem here proposed on the supposition that the obstacle is spherical, without any restriction as to the smallness of the change of mechanical properties; in the present investigation the form of the obstacle is arbitrary, but we assume that the squares and higher powers of the changes of mechanical properties may be omitted.

If ξ, η, ζ denote the displacements parallel to the axes of co-ordinates of the particle, whose equilibrium position is defined by x, y, z, and if σ be the normal density, and m the constant of compressibility so that $\delta p = ms$, the equations of motion are

$$\sigma \frac{d^2\xi}{dt^2} + \frac{d\,(ms)}{dx} = 0 \quad \text{.....................(1)},$$

and two similar equations in η and ζ. On the assumption that the whole motion is proportional to $e^{i\kappa at}$, where as usual $\kappa = 2\pi\lambda^{-1}$, and (§ 244) $a^2 = m\sigma^{-1}$, (1) may be written

$$\frac{d\,(ms)}{dx} - \sigma\kappa^2 a^2 \xi = 0 \quad \text{.....................(2)}.$$

The relation between the condensation s, and the displacements ξ, η, ζ, obtained by integrating (3) § 238 with respect to the time, is

$$-s = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} \quad\text{.....................(3).}$$

For the system of primary waves advancing in the direction of $-x$, η and ζ vanish; if ξ_0, s_0 be the values of ξ and s, and m_0, σ_0 be the mechanical constants for the undisturbed medium, we have as in (2)

$$\frac{d\,(m_0 s_0)}{dx} - \sigma_0 \kappa^2 a^2 \xi_0 = 0 \quad\text{..................(4);}$$

but ξ_0, s_0 do not satisfy (2) at the region of disturbance on account of the variation in m and σ, which occurs there. Let us assume that the complete values are $\xi_0 + \xi$, η, ζ, $s_0 + s$, and substitute in (2). Then taking account of (4), we get

$$\frac{d\,(ms)}{dx} - \sigma\kappa^2 a^2 \xi + (m - m_0)\frac{ds_0}{dx} + s_0 \frac{dm}{dx} - (\sigma - \sigma_0)\,\kappa^2 a^2 \xi_0 = 0,$$

or, as it may also be written,

$$\frac{d}{dx}(ms) - \sigma\kappa^2 a^2 \xi + \frac{d}{dx}(\Delta m . s_0) - \Delta\sigma . \kappa^2 a^2 \xi_0 = 0 \quad\text{........(5),}$$

if Δm, $\Delta\sigma$ stand respectively for $m - m_0$, $\sigma - \sigma_0$. The equations in η and ζ are in like manner

$$\left.\begin{aligned} \frac{d}{dy}(ms) - \sigma\kappa^2 a^2 \eta + \frac{d}{dy}(\Delta m . s_0) = 0 \\ \frac{d}{dz}(ms) - \sigma\kappa^2 a^2 \zeta + \frac{d}{dz}(\Delta m . s_0) = 0 \end{aligned}\right\} \quad\text{.........(6).}$$

It is to be observed that Δm, $\Delta\sigma$ vanish, except through a small space, which is regarded as the region of disturbance; ξ, η, ζ, s, being the result of the disturbance are to be treated as small quantities of the order Δm, $\Delta\sigma$; so that in our approximate analysis the variations of m and σ in the first two terms of (5) and (6) are to be neglected, being there multiplied by small quantities. We thus obtain from (5) and (6) by differentiation and addition, with use of (3), as the differential equation in s,

$$\nabla^2 (ms) + \kappa^2 ms = \kappa^2 a^2 \frac{d}{dx}(\Delta\sigma . \xi_0) - \nabla^2 (\Delta m . s_0) \quad\text{........(7).}$$

As in § 277, the solution of (7) is

$$4\pi m s = \iiint \frac{e^{-i\kappa r}}{r} \left\{ \nabla^2 (\Delta m . s_0) - \kappa^2 a^2 \frac{d}{dx} (\Delta\sigma . \xi_0) \right\} dV \text{......}(8),$$

in which the integration extends over a volume completely including the region of disturbance. The integrals in (8) may be transformed with the aid of Green's theorem. Calling the two parts respectively P and Q, we have

$$P = \iiint \frac{e^{-i\kappa r}}{r} \nabla^2 (\Delta m . s_0)\, dV = \iiint \Delta m . s_0 \nabla^2 \left(\frac{e^{-i\kappa r}}{r}\right) dV$$

$$+ \iint \left\{ \frac{e^{-i\kappa r}}{r} \frac{d}{dn} (\Delta m . s_0) - \Delta m . s_0 \frac{d}{dn} \left(\frac{e^{-i\kappa r}}{r}\right) \right\} dS,$$

where S denotes the surface of the space through which the triple integration extends. Now on S, Δm and $\frac{d}{dn}(\Delta m . s_0)$ vanish, so that both the surface integrals disappear. Moreover

$$\nabla^2 \left(\frac{e^{-i\kappa r}}{r}\right) = \frac{1}{r} \frac{d^2}{dr^2} e^{-i\kappa r} = -\kappa^2 \frac{e^{-i\kappa r}}{r};$$

and thus

$$P = -\kappa^2 \iiint \frac{e^{-i\kappa r}}{r} \Delta m . s_0\, dV \text{.................}(9).$$

If the region of disturbance be small in comparison with λ, we may write

$$P = -\kappa^2 s_0 \frac{e^{-i\kappa r}}{r} \iiint \Delta m\, dV \text{.................}(10).$$

In like manner for the second integral in (8), we find

$$Q = -\kappa^2 a^2 \iiint \frac{e^{-i\kappa r}}{r} \frac{d}{dx} (\Delta\sigma . \xi_0)\, dV$$

$$= \kappa^2 a^2 \iiint \Delta\sigma . \xi_0 \frac{d}{dx} \left(\frac{e^{-i\kappa r}}{r}\right) dV = i\kappa^3 a^2 \xi_0 \mu \frac{e^{-i\kappa r}}{r} \iiint \Delta\sigma\, dV \text{...}(11),$$

where μ denotes the cosine of the angle between x and r. The linear dimension of the region of disturbance is neglected in comparison with λ, and λ is neglected in comparison with r.

If T be the volume of the space through which Δm, $\Delta\sigma$ are sensible, we may write

$$\iiint \Delta m\, dV = T . \Delta m, \qquad \iiint \Delta\sigma\, dV = T . \Delta\sigma,$$

if on the right-hand sides Δm, $\Delta\sigma$ refer to the *mean values* of the variations in question. Thus from (8)

$$s = -\frac{\kappa^2 T e^{-i\kappa r}}{4\pi m r}\left\{\Delta m . s_0 - i\kappa a^2 \Delta\sigma . \xi_0 \mu\right\} \;\ldots\ldots(12).$$

To express ξ_0 in terms of s_0, we have from (3), $\xi_0 = -\int s_0\, dx$; and thus, if the condensation for the primary waves be $s_0 = e^{i\kappa(at+x)}$, $i\kappa\xi_0 = -s_0$, and (12) may be put into the form

$$s : s_0 = -\frac{\pi T e^{-i\kappa r}}{\lambda^2 r}\left\{\frac{\Delta m}{m} + \frac{\Delta\sigma}{\sigma}\mu\right\} \ldots\ldots\ldots\ldots(13),$$

in which s_0 denotes the condensation of the primary waves at the place of disturbance at time t, and s denotes the condensation of the secondary waves at the same time at a distance r from the disturbance. Since the difference of phase represented by the factor $e^{-i\kappa r}$ corresponds simply to the distance r, we may consider that a simple reversal of phase occurs at the place of disturbance. The amplitude of the secondary waves is inversely proportional to the distance r, and to the *square* of the wave-length λ. Of the two terms expressed in (13) the first is symmetrical in all directions round the place of disturbance, while the second varies as the cosine of the angle between the primary and the secondary rays. Thus a place at which m varies behaves as a *simple* source, and a place at which σ varies behaves as a *double* source (§ 294).

That the secondary disturbance must vary as λ^{-2} may be proved immediately by the method of dimensions. Δm and $\Delta\sigma$ being given, the amplitude is necessarily proportional to T, and in accordance with the principle of energy must also vary inversely as r. Now the only quantities (dependent upon space, time, and mass) of which the ratio of amplitudes can be a function, are T, r, λ, a (the velocity of sound), and σ, of which the last cannot occur in the expression of a simple ratio, as it is the only one of the five which involves a reference to mass. Of the remaining four quantities T, r, λ, and a, the last is the only one which involves a reference to time, and is therefore excluded. We are left with T, r, and λ, of which the only combination varying as Tr^{-1}, and independent of the unit of length, is $Tr^{-1}\lambda^{-2}$.[1]

An interesting application of the results of this section may be made to explain what have been called *harmonic echoes*[2].

[1] "On the light from the sky," *Phil. Mag.* Feb. 1871, and "On the scattering of light by small Particles," *Phil. Mag.* June, 1871.

[2] *Nature*, 1873, VIII. 319.

If the primary sound be a compound musical note, the various component tones are scattered in unlike proportions. The octave, for example, is sixteen times stronger relatively to the fundamental tone in the secondary than it was in the primary sound. There is thus no difficulty in understanding how it may happen that echoes returned from such reflecting bodies as groups of trees may be raised an octave. The phenomenon has also a complementary side. If a number of small bodies lie in the path of waves of sound, the vibrations which issue from them in all directions are at the expense of the energy of the main stream, and where the sound is compound, the exaltation of the higher harmonics in the scattered waves involves a proportional deficiency of them in the direct wave after passing the obstacles. This is perhaps the explanation of certain echoes which are said to return a sound graver than the original; for it is known that the pitch of a pure tone is apt to be estimated too low. But the evidence is conflicting, and the whole subject requires further careful experimental investigation; it may be commended to the attention of those who may have the necessary opportunities. While an alteration in the *character* of a sound is easily intelligible, and must indeed generally happen to a limited extent, a change in the pitch of a simple tone would be a violation of the law of forced vibrations, and hardly to be reconciled with theoretical ideas.

In obtaining (13) we have neglected the effect of the variable nature of the medium *on the disturbance*. When the disturbance on this supposition is thoroughly known, we might approximate again in the same manner. The additional terms so obtained would be necessarily of the second order in Δm, $\Delta\sigma$, so that our expressions are in all cases correct as far as the first powers of those quantities.

Even when the region of disturbance is not small in comparison with λ, the same method is applicable, provided the squares of Δm, $\Delta\sigma$ be really negligible. The total effect of any obstacle may then be calculated by integration from those of its parts. In this way we may trace the transition from a small region of disturbance whose *surface* does not come into consideration, to a thin plate of a few or of a great many square wave-lengths in area, which will ultimately reflect according to the regular optical law. But if the obstacle be at all elongated in the direction of the primary rays, this method of calculation soon

ceases to be practically available, because, even although the change of mechanical properties be very small, the interaction of the various parts of the obstacle cannot be left out of account. This caution is more especially needed in dealing with the case of light, where the wave-length is so exceedingly small in comparison with the dimensions of ordinary obstacles.

297. In some degree similar to the effect produced by a change in the mechanical properties of a small region of the fluid, is that which ensues when the square of the motion rises anywhere to such importance that it can be no longer neglected. $\nabla^2\phi + \kappa^2\phi$ then acquires a finite value dependent upon the square of the motion. Such places therefore act like sources of sound; the periods of the sources including the submultiples of the original period. Thus any part of space, at which the intensity accumulates to a sufficient extent, becomes itself a secondary source, emitting the harmonic tones of the primary sound. If there be two primary sounds of sufficient intensity, the secondary vibrations have frequencies which are the sums and differences of the frequencies of the primaries (§ 68)[1].

298. The pitch of a sound is liable to modification when the source and the recipient are in relative motion. It is clear, for instance, that an observer approaching a fixed source will meet the waves with a frequency exceeding that proper to the sound, by the number of wave-lengths passed over in a second of time. Thus if v be the velocity of the observer and a that of sound, the frequency is altered in the ratio $a \pm v : a$, according as the motion is towards or from the source. Since the alteration of pitch is constant, a musical performance would still be heard in tune, although in the second case, when a and v are nearly equal, the fall in pitch would be so great as to destroy all musical character. If we could suppose v to be greater than a, a sound produced after the motion had begun would never reach the observer, but sounds previously excited would be gradually overtaken and heard in the reverse of the natural order. If $v = 2a$, the observer would hear a musical piece in correct time and tune, but *backwards*.

Corresponding results ensue when the source is in motion and the observer at rest; the alteration depending only on the relative motion in the line of hearing. If the source and the observer move with the same velocity there is no alteration of frequency, whether

[1] Helmholtz über Combinationstöne. Pogg. *Ann.* Bd. xcix. s. 497. 1856.

the medium be in motion, or not. With a relative motion of 40 miles per hour the alteration of pitch is very conspicuous, amounting to about a semitone. The whistle of a locomotive is heard too high as it approaches, and too low as it recedes from an observer at a station, changing rather suddenly at the moment of passage.

The principle of the alteration of pitch by relative motion was first enunciated by Doppler[1], and is often called Doppler's principle. Strangely enough its legitimacy was disputed by Petzval[2], whose objection was the result of a confusion between two perfectly distinct cases, that in which there is a relative motion of the source and recipient, and that in which the medium is in motion while the source and the recipient are at rest. In the latter case the circumstances are mechanically the same as if the medium were at rest and the source and the recipient had a common motion, and therefore by Doppler's principle no change of pitch is to be expected.

Doppler's principle has been experimentally verified by Buijs Ballot[3] and Scott Russell, who examined the alterations of pitch of musical instruments carried on locomotives. A laboratory instrument for proving the change of pitch due to motion has been invented by Mach[4]. It consists of a tube six feet in length, capable of turning about an axis at its centre. At one end is placed a small whistle or reed, which is blown by wind forced along the axis of the tube. An observer situated in the plane of rotation hears a note of fluctuating pitch, but if he places himself in the prolongation of the axis of rotation, the sound becomes steady. Perhaps the simplest experiment is that described by König[5]. Two c'' tuning forks mounted on resonance cases are prepared to give with each other four beats per second. If the graver of the forks be made to approach the ear while the other remains at rest, one beat is *lost* for each two feet of approach; if, however, it be the more acute of the two forks which approaches the ear, one beat is *gained* in the same distance. A modification

[1] Theorie des farbigen Lichtes der Doppelsterne. Prag, 1842. See Pisko, *Die neueren Apparate der Akustik.* Wien, 1865.

[2] *Wien. Ber.* VIII. 134. 1852. *Fortschritte der Physik*, VIII. 167.

[3] Pogg. *Ann.* LXVI. p. 321.

[4] Pogg. *Ann.* CXII. p. 66, 1861, and CXVI. p. 333.

[5] König's *Catalogue des Appareils d'Acoustique.* Paris, 1865.

of this experiment due to Mayer[1] may also be noticed. In this case one fork excites the vibrations of a second in unison with itself, the excitation being made apparent by a small pendulum, whose bob rests against the extremity of one of the prongs. If the exciting fork be at rest, the effect is apparent up to a distance of 60 feet, but it ceases when the exciting fork is moved rapidly to or fro in the direction of the line joining the two forks.

There is some difficulty in treating mathematically the problem of a moving source, arising from the fact that any practical source acts also as an obstacle. Thus in the case of a bell carried through the air, we should require to solve a problem difficult enough without including the vibrations at all. But the solution of such a problem, even if it could be obtained, would throw no particular light on Doppler's law, and we may therefore advantageously simplify the question by idealizing the bell into a simple source of sound.

In § 147 we considered the problem of a moving source of disturbance in the case of a stretched string. The theory for aerial waves in one dimension is precisely similar, but for the general case of three dimensions some extension is necessary, in order to take account of the possibility of a motion across the direction of the sound rays. From §§ 273, 276 it appears that the effect at any point O of a source of sound is the same, whether the source be at rest, or whether it move in any manner on the surface of a sphere described about O as centre. If the source move in such a manner as to change its distance (r) from O, its effect is altered in two ways. Not only is the *phase* of the disturbance on arrival at O affected by the variation of distance, but the *amplitude* also undergoes a change. The latter complication however may be put out of account, if we limit ourselves to the case in which the source is sufficiently distant. On this understanding we may assert that the effect at O of a disturbance generated at time t and at distance r is the same as that of a similar disturbance generated at the time $t + \delta t$ and at the distance $r - a\,\delta t$. In the case of a periodic disturbance a velocity of approach (v) is equivalent to an increase of frequency in the ratio $a : a + v$.

299. We will now investigate the forced vibrations of the air contained within a rectangular chamber, due to internal sources of sound. By § 267 it appears that the result at time t of an

[1] *Phil. Mag.* (4) XLIII. p. 278, 1872.

initial condensation confined to the neighbourhood of the point ξ, η, ζ is

$$\dot{\phi} = \Sigma\Sigma\Sigma\, \kappa a B_{pqr} \cos \kappa a t \cos\left(p\frac{\pi x}{\alpha}\right)\cos\left(q\frac{\pi y}{\beta}\right)\cos\left(r\frac{\pi z}{\gamma}\right) \quad \ldots(1),$$

where

$$\kappa a B_{pqr} = \frac{8}{\alpha\beta\gamma}\cos\left(p\frac{\pi\xi}{\alpha}\right)\cos\left(q\frac{\pi\eta}{\beta}\right)\cos\left(r\frac{\pi\zeta}{\gamma}\right)\iiint \dot{\phi}_0\, dx\,dy\,dz \ldots(2),$$

from which the effect of an impressed force may be deduced, as in § 276. The disturbance $\iiint \dot{\phi}_{t'}\, dx\,dy\,dz$ communicated at time t' being denoted by $\iiint \Phi(t')\,dt'\,dx\,dy\,dz$, or $\Phi_1(t')\,dt'$, the resultant disturbance at time t is

$$\dot{\phi} = \frac{8}{\alpha\beta\gamma}\Sigma\Sigma\Sigma \cos\left(p\frac{\pi x}{\alpha}\right)\cos\left(q\frac{\pi y}{\beta}\right)\cos\left(r\frac{\pi z}{\gamma}\right)\times$$

$$\cos\left(p\frac{\pi\xi}{\alpha}\right)\cos\left(q\frac{\pi\eta}{\beta}\right)\cos\left(r\frac{\pi\zeta}{\gamma}\right)\int_{-\infty}^{t}\Phi_1(t')\cos\kappa a(t-t')\,dt' \ldots(3).$$

The symmetry of this expression with respect to x, y, z and ξ, η, ζ is an example of the principle of reciprocity (§ 107).

In the case of a harmonic force, for which $\Phi_1(t') = A\cos mat'$, we have to consider the value of

$$\int_{-\infty}^{t}\cos mat'\cos\kappa a(t-t')\,dt' \ldots\ldots\ldots\ldots\ldots(4).$$

Strictly speaking, this integral has no definite value; but, if we wish for the expression of the forced vibrations only, we must omit the integrated function at the lower limit, as may be seen by supposing the introduction of very small dissipative forces. We thus obtain

$$\int_{-\infty}^{t}\Phi_1(t')\cos\kappa a(t-t')\,dt' = A\frac{ma\sin mat}{(m^2-\kappa^2)a^2} \ldots\ldots(5).$$

As might have been predicted, the expressions become infinite in case of a coincidence between the period of the source and one of the natural periods of the chamber. Any particular normal vibration will not be excited, if the source be situated on one of its loops.

The effect of a multiplicity of sources may readily be inferred by summation or integration.

300. When sound is excited within a cylindrical pipe, the simplest kind of excitation that we can suppose is by the forced vibration of a piston. In this case the waves are plane from the beginning. But it is important also to inquire what happens when the source, instead of being uniformly diffused over the section, is concentrated in one point of it. If we assume (what, however, is not unreservedly true) that at a sufficient distance from the source the waves become plane, the law of reciprocity is sufficient to guide us to the desired information.

Let A be a simple source in an unlimited tube, B, B' two points of the same normal section in the region of plane waves. *Ex hypothesi*, the potentials at B and B' due to the source A are the same, and accordingly by the law of reciprocity equal sources at B and B' would give the same potential at A. From this it follows that the effect of any source is the same at a distance, as if the source were uniformly diffused over the section which passes through it. For example, if B and B' were equal sources in opposite phases, the disturbance at A would be nil.

The energy emitted by a simple source situated within a tube may now be calculated. If the section of the tube be σ, and the source such that in the open the potential due to it would be

$$\phi = -\frac{A}{4\pi} \cdot \frac{\cos \kappa (at - r)}{r} \qquad \text{..............(1)},$$

the velocity-potential at a distance within the tube will be the same as if the cause of the disturbance were the motion of a piston at the origin, giving the same total displacement, and the energy emitted will also be the same. Now from (1)

$$2\pi r^2 \frac{d\phi}{dr} = \tfrac{1}{2} A \cos \kappa at \text{ ultimately},$$

and therefore if ψ be the velocity-potential of the plane waves in the tube (supposed parallel to x), we may take

$$\sigma \frac{d\psi}{dx} = \tfrac{1}{2} A \cos \kappa (at - x) \qquad \text{..............(2)},$$

corresponding to which

$$\psi = -\frac{aA}{2\sigma} \cos \kappa (at - x) \qquad \text{..................(3)}.$$

Hence, as in § 245, the energy (W) emitted *on each side* of the source is given by

$$\frac{dW}{dt} = \sigma\left(-\rho\,\dot{\psi}\,\frac{d\psi}{dz}\right)_{z=0} = \frac{\rho a A^2}{4\sigma}\cos^2\kappa a t\,;$$

so that in the long run

$$W = \frac{\rho a A^2}{8\sigma}\,t\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If the tube be stopped by an immovable piston placed close to the source, the whole energy is emitted in one direction; but this is not all. In consequence of the doubled pressure, twice as much energy as before is developed, and thus in this case

$$W = \frac{\rho a A^2}{2\sigma}\,t\,\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The narrower the tube, the greater is the energy issuing from a given source. It is interesting to compare the efficiency of a source at the stopped end of a cylindrical tube with that of an equal source situated at the vertex of a cone. From § 280 we have in the latter case,

$$W' = \rho\,\frac{\kappa^2 a A^2}{2\omega}\,t\,\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

so that

$$W : W' = \omega : \kappa^2\sigma\,\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

The energies emitted in the two cases are the same when $\omega = \kappa^2\sigma$, that is, when the section of the cylinder is equal to the area cut off by the cone from a sphere of radius κ^{-1}.

301. We have now to examine how far it is true that vibrations within a cylindrical tube become approximately plane at a sufficient distance from their source. Taking the axis of z parallel to the generating lines of the cylinder, let us investigate the motion, whose potential varies as $e^{i\kappa a t}$, on the positive side of a source, situated at $z=0$. If ϕ be the potential and ∇^2 stand for $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}$, the equation of the motion is

$$\left(\frac{d^2}{dz^2}+\nabla^2+\kappa^2\right)\phi = 0\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If ϕ be independent of z, it represents vibrations wholly transverse to the axis of the cylinder. If the potential be then proportional to e^{ipat}, it must satisfy

$$(\nabla^2+p^2)\,\phi = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

as well as the condition that over the boundary of the section

$$\frac{d\phi}{dn} = 0 \qquad\qquad (3).$$

In order that these equations may be compatible, p is restricted to certain definite values corresponding to the periods of the natural vibrations. A zero value of p gives $\phi =$ constant, which solution, though it is of no significance in the two dimension problem, we shall presently have to consider. For each admissible value of p, there is a definite normal function u of x and y (§ 92), such that a solution is

$$\phi = A\, u\, e^{ipat} \qquad\qquad (4).$$

Two functions u, u', corresponding to different values of p, are conjugate, viz. make

$$\iint uu'\,dx\,dy = 0 \qquad\qquad (5),$$

and any function of x and y may be expanded within the contour in the series

$$\phi = A_0 u_0 + A_1 u_1 + A_2 u_2 + \ldots \qquad\qquad (6),$$

in which u_0, corresponding to $p = 0$, is constant.

In the actual problem ϕ may still be expanded in the same series, provided that A_0, A_1, &c. be regarded as functions of z. By substitution in (1) we get, having regard to (2),

$$u_0\left\{\frac{d^2A_0}{dz^2} + \kappa^2 A_0\right\} + u_1\left\{\frac{d^2A_1}{dz^2} + (\kappa^2 - p_1^2)A_1\right\}$$
$$+ u_2\left\{\frac{d^2A_2}{dz^2} + (\kappa^2 - p_2^2)A_2\right\} + \ldots = 0 \qquad\qquad (7),$$

in which, by virtue of the conjugate property of the normal functions, each coefficient of u must vanish separately. Thus

$$\frac{d^2A_0}{dz^2} + \kappa^2 A_0 = 0, \qquad \frac{d^2A}{dz^2} + (\kappa^2 - p^2)A = 0 \qquad\qquad (8).$$

The solution of the first of these equations is

$$A_0 = \alpha_0 e^{i\kappa z} + \beta_0 e^{-i\kappa z},$$

giving

$$\phi_0 = \alpha_0 u_0 e^{i\kappa(at+z)} + \beta_0 u_0 e^{i\kappa(at-z)} \qquad\qquad (9).$$

The solution of the general equation in A assumes a different form, according as $\kappa^2 - p^2$ is positive or negative. If the forced

vibration be graver in pitch than the gravest of the purely transverse natural vibrations, every finite value of p^2 is greater than κ^2, or $\kappa^2 - p^2$ is always negative. Putting

$$\kappa^2 - p^2 = -\mu^2 \qquad\qquad (10),$$

we have
$$A = \alpha e^{\mu z} + \beta e^{-\mu z},$$

whence
$$\phi = (\alpha e^{\mu z} + \beta e^{-\mu z})\, u e^{i\kappa at} \qquad\qquad (11).$$

Now under the circumstances supposed, it is evident that the motion does not become infinite with z, so that all the coefficients α vanish. For a somewhat different reason the same is true of α_0, as there can be no wave in the negative direction. We may therefore take

$$\phi = \beta_0 e^{i\kappa(at-z)} + \beta_1 u_1 e^{-\mu_1 z} e^{i\kappa at} + \beta_2 u_2 e^{-\mu_2 z} e^{i\kappa at} + \ldots \qquad\qquad (12),$$

an expression which reduces to its first term when z is sufficiently great. We conclude that in all cases the waves ultimately become plane, *if the forced vibration be graver than the gravest of the natural transverse vibrations.*

In the case of a circular cylinder, of radius r, the gravest transverse vibration has a wave-length equal to $2\pi r \div 1{\cdot}841 = 3{\cdot}413r$ (§ 339). If then the wave-length of the forced vibration exceed $3{\cdot}413r$, the waves ultimately become plane. It may happen however that the waves ultimately become plane, although the wave-length fall short of the above limit. For example, if the source of vibration be symmetrical with respect to the axis of the tube, *e.g.* a simple source situated on the axis itself, the gravest transverse vibration with which we should have to deal would be more than an octave higher than in the general case, and the wave-length of the forced vibration might have less than half the above value.

From (12), when $z = 0$,

$$\frac{d\phi}{dz} = -i\kappa\beta_0 e^{i\kappa at} - \mu_1\beta_1 u_1 e^{i\kappa at} - \ldots$$

whence

$$\iint \frac{d\phi}{dz}\, d\sigma = -i\kappa\beta_0 \sigma e^{i\kappa at} \qquad\qquad (13),$$

inasmuch as $\iint u_1\, d\sigma$, $\iint u_2\, d\sigma$, &c., all vanish.

It appears accordingly that the plane waves at a distance are the same as would be produced by a rigid piston at the origin,

giving the same mean normal velocity as actually exists. Any normal motion of which the negative and positive parts are equal, produces ultimately no effect.

When there is no restriction on the character of the source, and when some of the transverse natural vibrations are graver than the actual one, some of the values of $\kappa^2 - p^2$ are positive, and then terms enter of the form

$$\phi = \beta u\, e^{i\kappa at}\, e^{-i\sqrt{(\kappa^2-p^2)}z},$$

or in real quantities

$$\phi = \beta u \cos\{\kappa at - \sqrt{(\kappa^2 - p^2)}\, z\} \quad\text{..................}(14),$$

indicating that the peculiarities of the source are propagated to an infinite distance.

The problem here considered may be regarded as a generalization of that of § 268. For the case of a circular cylinder it may be worked out completely with the aid of Bessel's functions, but this must be left to the reader.

302. In § 278 we have fully determined the motion of the air due to the normal periodic motion of a bounding plane plate of infinite extent. If $\frac{d\phi}{dn}$ be the given normal velocity at the element dS,

$$\phi = -\frac{1}{2\pi}\iint \frac{d\phi}{dn}\frac{e^{-i\kappa r}}{r}\, dS \quad\text{.....................}(1)$$

gives the velocity-potential at any point P distant r from dS. The remainder of this chapter is devoted to the examination of the particular case of this problem which arises when the normal velocity has a given constant value over a circular area of radius R, while over the remainder of the plane it is zero. In particular we shall investigate what forces due to the reaction of the air will act on a rigid circular plate, vibrating with a simple harmonic motion in an equal circular aperture cut out of a rigid plane plate extending to infinity.

For the whole variation of pressure acting on the plate we have (§ 244)

$$\iint \delta p\, dS = -\sigma \iint \dot{\phi}\, dS = -i\kappa a\sigma \iint \phi\, dS,$$

where σ is the natural density, and ϕ varies as $e^{i\kappa at}$. Thus by (1)

$$\iint \delta p\, dS = \frac{i\kappa a\sigma}{\pi}\frac{d\phi}{dn}\Sigma\Sigma\frac{e^{-i\kappa r}}{r}\, dS\, dS' \text{.............} (2).$$

In the double sum

$$\Sigma\Sigma\frac{e^{-i\kappa r}}{r}\, dS\, dS' \text{........................} (3),$$

which we have now to evaluate, each pair of elements is to be taken *once* only, and the product is to be summed after multiplication by the factor $r^{-1}e^{-i\kappa r}$, depending on their mutual distance. The best method is that suggested by Prof. Maxwell for the common potential[1]. The quantity (3) is regarded as the work that would be consumed in the complete dissociation of the matter composing the disc, that is to say, in the removal of every element from the influence of every other, on the supposition that the potential of two elements is proportional to $r^{-1}e^{-i\kappa r}$. The amount of work required, which depends only on the initial and final states, may be calculated by supposing the operation performed in any way that may be most convenient. For this purpose we suppose that the disc is divided into elementary rings, and that each ring is carried away to infinity before any of the interior rings are disturbed.

The first step is the calculation of the potential (V) at the edge of a disc of radius c. Taking polar co-ordinates (ρ, θ) with any point of the circumference for pole, we have

$$V = \iint \frac{e^{-i\kappa\rho}}{\rho}\rho\, d\rho\, d\theta = \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{2c\cos\theta} e^{-i\kappa\rho}\, d\rho\, d\theta = \frac{2}{i\kappa}\int_0^{\frac{1}{2}\pi}\{1 - e^{-2i\kappa c\cos\theta}\}\, d\theta.$$

This quantity must be multiplied by $2\pi c\, dc$, and afterwards integrated with respect to c between the limits 0 and R. But it will be convenient first to effect a transformation. We have

$$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} e^{-2i\kappa c\cos\theta}\, d\theta = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} e^{-2i\kappa c\sin\theta}\, d\theta$$

$$= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\cos(2\kappa c\sin\theta)\, d\theta - \frac{2i}{\pi}\int_0^{\frac{1}{2}\pi}\sin(2\kappa c\sin\theta)\, d\theta$$

$$= J_0(z) - iK(z) \text{..} (4),$$

where z is written for $2\kappa c$. $J_0(z)$ is the Bessel's function of zero

[1] Theory of Resonance. *Phil. Trans.* 1870.

order (§ 200), and $K(z)$ is a function defined by the equation

$$K(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \sin(z\sin\theta)\, d\theta$$

$$= \frac{2}{\pi}\left\{z - \frac{z^3}{1^2 . 3^2} + \frac{z^5}{1^2 . 3^2 . 5^2} - \frac{z^7}{1^2 . 3^2 . 5^2 . 7^2} + \ldots\ldots\right\} \ldots\ldots\ldots(5).$$

Deferring for the moment the further consideration of the function K, we have

$$V = \frac{\pi}{\kappa}\left[K(z) - i\{1 - J_0(z)\}\right] \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and thus

$$\Sigma\Sigma\frac{e^{-i\kappa r}}{r}\, dS\, dS' = \frac{\pi^2}{2\kappa^3}\int_0^{2\kappa R} z\, dz\left[K(z) - i\{1 - J_0(z)\}\right].$$

Now by (6) § 200 and (8) § 204

$$\int_0^z z\, dz\, J_0(z) = z J_1(z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and thus, if K_1 be defined by

$$K_1(z) = \int_0^z z\, dz\, K(z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

we may write

$$\Sigma\Sigma\frac{e^{-i\kappa r}}{r}\, dS\, dS' = \frac{\pi^2}{2\kappa^3} K_1(2\kappa R) - i\frac{\pi^2 R^2}{\kappa}\left(1 - \frac{J_1(2\kappa R)}{\kappa R}\right). \ldots(9).$$

From this the total pressure is derived by introduction of the factor $-\dfrac{i\kappa a\sigma}{\pi}\dfrac{d\phi}{dn}$, so that

$$\iint \delta p\, dS = a\sigma . \pi R^2 . \frac{d\phi}{dn}\left(1 - \frac{J_1(2\kappa R)}{\kappa R}\right) + i\frac{a\sigma\pi}{2\kappa^2}\frac{d\phi}{dn} K_1(2\kappa R) .. (10).$$

The reaction of the air on the disc may thus be divided into two parts, of which the first is proportional to the velocity of the disc, and the second to the acceleration. If ξ denote the displacement of the disc, so that $\dot\xi = \dfrac{d\phi}{dn}$, we have $\ddot\xi = i\kappa a\,\dot\xi = i\kappa a\dfrac{d\phi}{dn}$; and therefore in the equation of motion of the disc, the reaction of the air is represented by a frictional force $a\sigma . \pi R^2 . \dot\xi\left(1 - \dfrac{J_1(2\kappa R)}{\kappa R}\right)$ retarding the motion, and by an accession to the inertia equal to $\dfrac{\pi\sigma}{2\kappa^3} K_1(2\kappa R)$.

When κR is small, we have from the ascending series for J_1 (5) § 200,

$$1 - \frac{J_1(2\kappa R)}{\kappa R} = \frac{\kappa^2 R^2}{1 \,.\, 2} - \frac{\kappa^4 R^4}{1 \,.\, 2^2 \,.\, 3} + \frac{\kappa^6 R^6}{1 \,.\, 2^2 \,.\, 3^2 \,.\, 4} - \frac{\kappa^8 R^8}{1 \,.\, 2^2 \,.\, 3^2 \,.\, 4^2 \,.\, 5} + \dots \quad (11),$$

so that the frictional term is approximately

$$\tfrac{1}{2}\, a\sigma \,.\, \pi R^2 \,.\, \kappa^2 R^2 \,.\, \dot{\xi} \quad \dots\dots\dots\dots(12).$$

From the nature of the case the coefficient of $\dot{\xi}$ must be positive, otherwise the reaction of the air would tend to augment, instead of to retard, the motion. That $J_1(z)$ is in fact always less than $\frac{1}{2}z$ may be verified as follows. If θ lie between 0 and π, and z be positive, $\sin(z \sin\theta) - z\sin\theta$ is negative, and therefore also

$$\frac{1}{\pi}\int_0^\pi \{\sin(z\sin\theta) - z\sin\theta\}\sin\theta \, d\theta$$

is negative. But this integral is $J_1(z) - \frac{1}{2}z$, which is accordingly negative for all positive values of z.

When κR is great, $J_1(2\kappa R)$ tends to vanish, and then the frictional term becomes simply $a\sigma \,.\, \pi R^2 \,.\, \dot{\xi}$. This result might have been expected; for when κR is very large, the wave motion in the neighbourhood of the disc becomes approximately plane. We have then by (6) and (8) § 245, $dp = a\rho_0 \dot{\xi}$, in which ρ_0 is the density (σ); so that the retarding force is $\pi R^2 \delta p = a\sigma . \pi R^2 . \dot{\xi}$.

We have now to consider the term representing an alteration of inertia, and among other things to prove that this alteration is an increase, or that $K_1(z)$ is positive. By direct integration of the ascending series (5) for K (which is always convergent),

$$K_1(z) = \frac{2}{\pi}\left\{\frac{z^3}{1^2 \,.\, 3} - \frac{z^5}{1^2 \,.\, 3^2 \,.\, 5} + \frac{z^7}{1^2 \,.\, 3^2 \,.\, 5^2 \,.\, 7} - \dots\dots\right\} \quad \dots\dots\dots(13).$$

When therefore κR is small, we may take as the expression for the increase of inertia

$$\frac{8\sigma R^3}{3} = \sigma \,.\, \pi R^2 \,.\, \frac{8R}{3\pi} \quad \dots\dots\dots\dots(14).$$

This part of the reaction of the air is therefore represented by supposing the vibrating plate to carry with it a mass of air equal to that contained in a cylinder whose base is the plate, and whose height is equal to $\frac{8R}{3\pi}$; so that, when the plate is sufficiently small, the mass to be added is independent of the period of vibration.

From the series (5) for $K(z)$, it may be proved immediately that

$$\frac{1}{z}\frac{d}{dz}\left(z\frac{d}{dz}\right)K(z)=\frac{2}{\pi z}-K(z)\quad\text{.................}(15),$$

or

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+1\right)K(z)=\frac{2}{\pi z}\quad\text{.................}(16).$$

From the first form (15) it follows that

$$K_1(z)=\int_0^z K(z)\,z\,dz=\frac{2}{\pi}z-z\frac{dK(z)}{dz}\quad\text{..........}(17).$$

By means of this expression for $K_1(z)$ we may readily prove that the function is always positive. For

$$\frac{dK(z)}{dz}=\frac{d}{dz}\cdot\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\sin(z\sin\theta)\,d\theta=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\cos(z\sin\theta)\sin\theta\,d\theta\text{...}(18);$$

so that

$$K_1(z)=\frac{2z}{\pi}\left\{1-\int_0^{\frac{1}{2}\pi}\cos(z\sin\theta)\sin\theta\,d\theta\right\}$$

$$=\frac{4z}{\pi}\int_0^{\frac{1}{2}\pi}\sin^2(\tfrac{1}{2}z\sin\theta)\sin\theta\,d\theta\quad\text{..............}(19),$$

an integral of which every element is positive. When z is very large, $\cos(z\sin\theta)$ fluctuates with great rapidity, and thus $K_1(z)$ tends to the form

$$K_1(z)=\frac{2}{\pi}\cdot z\quad\text{.........................}(20).$$

When z is great, the ascending series for K and K_1, though always ultimately convergent, become useless for practical calculation, and it is necessary to resort to other processes. It will be observed that the differential equation (16) satisfied by K is the same as that belonging to the Bessel's function J_0, with the exception of the term on the right-hand side, viz. $\dfrac{2}{\pi z}$. The function K is therefore included in the form obtained by adding to the general solution of Bessel's equation containing two arbitrary constants any particular solution of (16). Such a particular solution is

$$\tfrac{1}{2}\pi.K(z)=z^{-1}-z^{-3}+1^2.3^2.z^{-5}-1^2.3^2.5^2.z^{-7}+1^2.3^2.5^2.7^2.z^{-9}-\text{...}(21),$$

as may be readily verified on substitution. The series on the right of (21) notwithstanding its ultimate divergency, may be used successfully for computation when z is great. It is in fact

the analytical equivalent of $\int_0^\infty e^{-\beta}(z^2+\beta^2)^{-\frac{1}{2}}d\beta$, and we might take

$$K(z)=\frac{2}{\pi}\int_0^\infty\frac{e^{-\beta}d\beta}{\sqrt{z^2+\beta^2}}+\text{Complementary Function},$$

determining the two arbitrary constants by an examination of the forms assumed when z is very great. But it is perhaps simpler to follow the method used by Lipschitz[1] for Bessel's functions.

By (4) we have

$$J_0(z)-iK(z)=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}e^{-iz\cos\theta}d\theta=\frac{2}{\pi}\int_0^1\frac{e^{-izv}dv}{\sqrt{1-v^2}}\text{ ....(22).}$$

Consider the integral $\int\frac{e^{-zw}dw}{\sqrt{1+w^2}}$, where w is a complex variable of the form $u+iv$. Representing, as usual, simultaneous pairs of values of u and v by the co-ordinates of a point, we see that the value of the integral will be zero, if the integration with respect to w range round the rectangle, whose angular points are respectively 0, h, $h+i$, i, where h is any real positive quantity. Thus

$$\int_0^h\frac{e^{-zu}du}{\sqrt{1+u^2}}+\int_0^i\frac{e^{-z(h+iv)}d(iv)}{\sqrt{1+(h+iv)^2}}+\int_h^0\frac{e^{-z(u+i)}du}{\sqrt{1+(u+i)^2}}+\int_i^0\frac{e^{-izv}d(iv)}{\sqrt{1-v^2}}=0,$$

from which, if we suppose that $h=\infty$,

$$\int_0^1\frac{e^{-izv}dv}{\sqrt{1-v^2}}=-i\int_0^\infty\frac{e^{-zu}du}{\sqrt{1+u^2}}+i\int_0^\infty\frac{e^{-z(u+i)}du}{\sqrt{1+(u+i)^2}}\text{ ....(23).}$$

Replacing uz by β, we may write (23) in the form

$$\int_0^1\frac{e^{-izv}dv}{\sqrt{1-v^2}}=-i\int_0^\infty\frac{e^{-\beta}d\beta}{z\sqrt{1+\frac{\beta^2}{z^2}}}+\frac{e^{-i(z-\frac{1}{4}\pi)}}{\sqrt{2z}}\int_0^\infty\frac{e^{-\beta}\beta^{-\frac{1}{2}}d\beta}{\sqrt{1-\frac{i\beta}{2z}}}\text{ ......(24).}$$

The first term on the right in (24) is entirely imaginary; it therefore follows by (22) that $\frac{1}{2}\pi J_0(z)$ is the real part of the second term. By expanding the binomial under the integral sign, and afterwards integrating by the formula

$$\int_0^\infty e^{-\beta}\beta^{q-\frac{1}{2}}d\beta=\Gamma(q+\tfrac{1}{2}),$$

we obtain as the expansion for $J_0(z)$ in negative powers of z,

$$J_0(z)=\sqrt{\left(\frac{2}{\pi z}\right)}\left\{1-\frac{1^2.3^2}{1.2.(8z)^2}+\dots\right\}\cos(z-\tfrac{1}{4}\pi)$$
$$+\sqrt{\left(\frac{2}{\pi z}\right)}\left\{\frac{1^2}{1.8z}-\frac{1^2.3^2.5^2}{1.2.3.(8z)^3}+\dots\right\}\sin(z-\tfrac{1}{4}\pi)\text{ ......(25).}$$

[1] Crelle, Bd. LVI. 1859. Lömmel, *Studien über die Bessel'schen Functionen*, p. 59.

By stopping the expansion after any desired number of terms, and forming the expression for the remainder, it may be proved that the error committed by neglecting the remainder cannot exceed the last term retained (§ 200).

In like manner the imaginary part of the right-hand member of (24) is the equivalent of $-\frac{1}{2}i\pi K(z)$, so that

$$K(z)=\frac{2}{\pi}\left\{z^{-1}-z^{-3}+1^2.3^2.z^{-5}-1^2.3^2.5^2.z^{-7}+\ldots\ldots\right\}$$

$$+\sqrt{\left(\frac{2}{\pi z}\right)}\left\{1-\frac{1^2.3^2}{1.2.(8z)^2}+\frac{1^2.3^2.5^2.7^2}{1.2.3.4.(8z)^4}-\ldots\ldots\right\}\sin\left(z-\tfrac{1}{4}\pi\right)$$

$$-\sqrt{\left(\frac{2}{\pi z}\right)}\left\{\frac{1^2}{1.8z}-\frac{1^2.3^2.5^2}{1.2.3.(8z)^3}+\ldots\ldots\right\}\cos\left(z-\tfrac{1}{4}\pi\right)\ \ldots\ldots\ldots(26).$$

The value of $K_1(z)$ may now be determined by means of (17). We find

$$\frac{dK}{dz}=-\frac{2}{\pi}\{z^{-2}-3.z^{-4}+1^2.3^2.5.z^{-6}-1^2.3^2.5^2.7.z^{-8}+\ldots\ldots\}$$

$$+\sqrt{\left(\frac{2}{\pi z}\right)}\cos\left(z-\tfrac{1}{4}\pi\right)\left\{1+\frac{3.5.1}{1.2.(8z)^2}-\frac{3.5.7.9.1.3.5}{1.2.3.4.(8z)^4}+\ldots\ldots\right\}$$

$$-\sqrt{\left(\frac{2}{\pi z}\right)}\sin\left(z-\tfrac{1}{4}\pi\right)\left\{\frac{3}{1.(8z)}-\frac{3.5.7.1.3}{1.2.3.(8z)^3}\right.$$

$$\left.+\frac{3.5.7.9.11.1.3.5.7}{1.2.3.4.5.(8z)^5}-\ldots\ldots\right\}\ldots\ldots(27).$$

The final expression for $K_1(z)$ may be put into the form

$$K_1(z)=\frac{2}{\pi}\{z+z^{-1}-3.z^{-3}+1^2.3^2.5.z^{-5}-1^2.3^2.5^2.7.z^{-7}+\ldots\ldots\}$$

$$-\sqrt{\frac{2z}{\pi}}.\cos\left(z-\tfrac{1}{4}\pi\right)\left\{1-\frac{(1^2-4)(3^2-4)}{1.2.(8z)^2}\right.$$

$$\left.+\frac{(1^2-4)(3^2-4)(5^2-4)(7^2-4)}{1.2.3.4.(8z)^4}-\ldots\ldots\right\}$$

$$-\sqrt{\frac{2z}{\pi}}.\sin\left(z-\tfrac{1}{4}\pi\right)\left\{\frac{1^2-4}{1.8z}-\frac{(1^2-4)(3^2-4)(5^2-4)}{1.2.3\,(8z)^3}+\ldots\ldots\right\}^{1}\ \ldots(28).$$

It appears then that K_1 does not vanish when z is great, but approximates to $\frac{2}{\pi}.z$. But although the accession to the inertia,

[1] As was to be expected, the series within brackets are the same as those that occur in the expression of the function $J_1(z)$.

which is proportional to K_1, becomes infinite with R, it vanishes ultimately when compared with the area of the disc, and with the other term which represents the dissipation. And this agrees with what we should anticipate from the theory of plane waves.

If, independently of the reaction of the air, the mass of the plate be M, and the force of restitution be $\mu\xi$, the equation of motion of the plate when acted on by an impressed force F, proportional to $e^{i\kappa at}$, will be

$$\left\{M + \frac{\pi\sigma}{2\kappa^3} K_1(2\kappa R)\right\}\ddot{\xi} + a\sigma\pi R^2\left(1 - \frac{J_1(2\kappa R)}{\kappa R}\right)\dot{\xi} + \mu\,\xi = F \text{ ...(29)};$$

or by (13), if, as will be usual in practical applications, κR be small,

$$\left(M + \frac{8\sigma R^3}{3}\right)\ddot{\xi} + \frac{a\sigma\pi\kappa^2 R^4}{2}\dot{\xi} + \mu\,\xi = F \text{..........(30)}.$$

Two particular cases of this problem deserve notice. First let M and μ vanish, so that the plate, itself devoid of mass, is subject to no other forces than F and those arising from aerial pressures. Since $\ddot{\xi} = i\kappa a\dot{\xi}$, the frictional term is relatively negligible, and we get when κR is very small,

$$a\sigma\pi R^2 \cdot \frac{8\kappa R}{3\pi}\dot{\xi} = -iF \text{ ....(31)}.$$

Next let M and μ be such that the natural period of the plate, when subject to the reaction of the air, is the same as that imposed upon it. Under these circumstances

$$\left(M + \frac{8\sigma R^3}{3}\right)\ddot{\xi} + \mu\xi = 0,$$

and therefore

$$a\sigma\pi R^2 \cdot \frac{\kappa^2 R^2}{2} \cdot \dot{\xi} = F \text{ ....(32)}.$$

Comparing with (31), we see that the amplitude of vibration is greater in the case when the inertia of the air is balanced, in the ratio of $16 : 3\pi\kappa R$, shewing a large increase when κR is small. In the first case the phase of the motion is such that comparatively very little work is done by the force F; while in the second, the inertia of the air is compensated by the spring, and then F, being of the same phase as the velocity, does the maximum amount of work.

CHAPTER XVI.

THEORY OF RESONATORS.

303. In the pipe closed at one end and open at the other we had an example of a mass of air endowed with the property of vibrating in certain definite periods peculiar to itself in more or less complete independence of the external atmosphere. If the air beyond the open end were entirely without mass, the motion within the pipe would have no tendency to escape, and the contained column of air would behave like any other complex system not subject to dissipation. In actual experiment the inertia of the external air cannot, of course, be got rid of, but when the diameter of the pipe is small, the effect produced in the course of a few periods may be insignificant, and then vibrations once excited in the pipe have a certain degree of persistence. The narrower the channel of communication between the interior of a vessel and the external medium, the greater does the independence become. Such cavities constitute resonators; in the presence of an external source of sound, the contained air vibrates in unison, and with an amplitude dependent upon the relative magnitudes of the natural and forced periods, rising to great intensity in the case of approximate isochronism. When the original cause of sound ceases, the resonator yields back the vibrations stored up as it were within it, thus becoming itself for a short time a secondary source. The theory of resonators constitutes an important branch of our subject.

As an introduction to it we may take the simple case of a stopped cylinder, in which a piston moves without friction. On the further side of the piston the air is supposed to be devoid of inertia, so that the pressure is absolutely constant. If now the piston be set into vibration of very long period, it is clear that the contained air will be at any time very nearly in the equilibrium condition (of uniform density) corresponding to the

momentary position of the piston. If the mass of the piston be very considerable in comparison with that of the included air, the natural vibrations resulting from a displacement will occur nearly as if the air had no inertia; and in deriving the period from the kinetic and potential energies, the former may be calculated without allowance for the inertia of the air, and the latter as if the rarefaction and condensation were uniform. Under the circumstances contemplated the air acts merely as a spring in virtue of its resistance to compression or dilatation; the form of the containing vessel is therefore immaterial, and the period of vibration remains the same, provided the capacity be not varied.

When a gas is compressed or rarefied, the mechanical value of the resulting displacement is found by multiplying each infinitesimal increment of volume by the corresponding pressure and integrating over the range required. In the present case it is of course only the difference of pressure on the two sides of the piston which is really operative, and this for a small change is proportional to the alteration of volume. The whole mechanical value of the small change is the same as if the expansion were opposed throughout by the *mean*, that is half the final, pressure; thus corresponding to a change of volume from S to $S+\delta S$, since $p=a^2\rho$,

$$V=p\cdot\frac{\delta S}{2S}\cdot\delta S=\tfrac{1}{2}\rho a^2\frac{(\delta S)^2}{S}\;\ldots\ldots\ldots\ldots\ldots\ldots(1)^1.$$

If A denote the area of the piston, M its mass, and x its linear displacement, $\delta S=Ax$, and the equation of motion is

$$M\ddot{x}+\frac{\rho a^2A^2}{S}x=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

indicating vibrations, whose periodic time is

$$\tau=2\pi\div aA\sqrt{\frac{\rho}{MS}}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Let us now imagine a vessel containing air, whose interior communicates with the external atmosphere by a narrow aperture or neck. It is not difficult to see that this system is capable of vibrations similar to those just considered, the air in the neighbourhood of the aperture supplying the place of the piston. By sufficiently increasing S, the period of the vibration may be made as long as we please, and we obtain finally a state of things in

[1] Compare (12) § 245.

which the kinetic energy of the motion may be neglected except in the neighbourhood of the aperture, and the potential energy may be calculated as if the density in the interior of the vessel were uniform. In flowing through the aperture under the operation of a difference of pressure on the two sides, or in virtue of its own inertia after such pressure has ceased, the air moves approximately as an incompressible fluid would do under like circumstances, provided that the space through which the kinetic energy is sensible be very small in comparison with the length of the wave. The suppositions on which we are about to proceed are not of course strictly correct as applied to actual resonators such as are used in experiment, but they are near enough to the mark to afford an instructive view of the subject and in many cases a foundation for a sufficiently accurate calculation of the pitch. They become rigorous only in the limit when the wave-length is indefinitely great in comparison with the dimensions of the vessel.

304. The kinetic energy of the motion of an incompressible fluid through a given channel may be expressed in terms of the density ρ, and the rate of transfer, or current, $\dot{X}$, for under the circumstances contemplated the character of the motion is always the same. Since T necessarily varies as ρ and as $\dot{X}^2$, we may put

$$T = \tfrac{1}{2}\rho\frac{\dot{X}^2}{c} \quad\text{.........................}\quad (1),$$

where the constant c, which depends only on the nature of the channel, is a linear quantity, as may be inferred from the fact that the dimensions of $\dot{X}$ are 3 in space and -1 in time. In fact, if ϕ be the velocity-potential,

$$T = \tfrac{1}{2}\rho\iiint\left[\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right]dx\,dy\,dz = \tfrac{1}{2}\rho\iint\phi\,\frac{d\phi}{dn}\,dS,$$

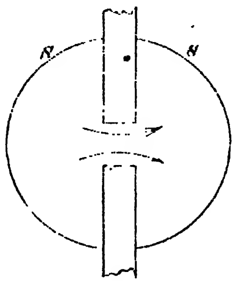
by Green's theorem, where the integration is to be extended over a surface including the whole region through which the motion is sensible. At a sufficient distance on either side of the aperture, ϕ becomes constant, and if the constant values be denoted by ϕ_1 and ϕ_2, and the integration be now limited to that half of S towards which the fluid flows, we have

$$T = \tfrac{1}{2}\rho\,(\phi_1 - \phi_2)\iint\frac{d\phi}{dn}\,dS = \tfrac{1}{2}\rho\,(\phi_1 - \phi_2)\,\dot{X}.$$

Now, since within S ϕ is determined linearly by its surface values, $\iint \frac{d\phi}{dn}\, dS$, or $\dot{X}$, is proportional to $(\phi_1 - \phi_2)$. If we put $\dot{X} = c\,(\phi_1 - \phi_2)$, we get as before

$$T = \tfrac{1}{2}\rho\,\frac{\dot{X}^2}{c}\,;$$

Fig. 58.



The nature of the constant c will be better understood by considering the electrical problem, whose conditions are mathematically identical with those of that under discussion. Let us suppose that the fluid is replaced by uniformly conducting material, and that the boundary of the channel or aperture is replaced by insulators. We know that if by battery power or otherwise, a difference of electric potential be maintained on the two sides, a steady current through the aperture of proportional magnitude will be generated. The ratio of the total current to the electromotive force is called the *conductivity* of the channel, and thus we see that our constant c represents simply this conductivity, on the supposition that the specific conducting power of the hypothetical substance is unity. The same thing may be otherwise expressed by saying that c is the side of the cube, whose resistance between opposite faces is the same as that of the channel. In the sequel we shall often avail ourselves of the electrical analogy.

When c is known, the proper tone of the resonator can be easily deduced. Since

$$V = \tfrac{1}{2}\rho a^2\,\frac{X^2}{S}\,,\qquad T = \tfrac{1}{2}\rho\,\frac{\dot{X}^2}{c}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

the equation of motion is

$$\ddot{X} + \frac{a^2 c}{S} X = 0 \text{ .... } (3),$$

indicating simple oscillations performed in a time

$$\tau = 2\pi \div \sqrt{\frac{a^2 c}{S}} \text{ .... } (4).$$

If N be the frequency, or number of complete vibrations executed in the unit time,

$$N = \frac{a}{2\pi}\sqrt{\frac{c}{S}} \text{ .... } (5).$$

The wave-length λ, which is the quantity most closely connected with the dimensions of the cavity, is given by

$$\lambda = \frac{a}{N} = 2\pi\sqrt{\frac{S}{c}} \text{ .... } (6),$$

and varies directly as the linear dimension. The wave-length, it will be observed, is a function of the size and shape of the resonator only, while the frequency depends also upon the nature of the gas; and it is important to remark that it is on the nature of the gas in and near the channel that the pitch depends and not on that occupying the interior of the vessel, for the inertia of the air in the latter situation does not come into play, while the compressibility of all gases is very approximately the same. Thus in the case of a pipe, the substitution of hydrogen for air in the neighbourhood of a node would make but little difference, but its effect in the neighbourhood of a loop would be considerable.

Hitherto we have spoken of the channel of communication as single, but if there be more than one channel, the problem is not essentially altered. The same formula for the frequency is still applicable, if as before we understand by c the whole conductivity between the interior and exterior of the vessel. When the channels are situated sufficiently far apart to act independently one of another, the resultant conductivity is the simple sum of those belonging to the separate channels; otherwise the resultant is less than that calculated by mere addition.

If there be two precisely similar channels, which do not interfere, and whose conductivity taken separately is c, we have

$$N = \sqrt{2} \times \frac{a}{2\pi}\sqrt{\frac{c}{S}} \text{ .... } (7),$$

shewing that the note is higher than if there were only one channel in the ratio $\sqrt{2} : 1$, or by rather less than a fifth—a law observed by Sondhauss and proved theoretically by Helmholtz in the case, where the channels of communication consist of simple holes in the infinitely thin sides of the reservoir.

305. The investigation of the conductivity for various kinds of channels is an important part of the theory of resonators; but in all except a very few cases the accurate solution of the problem is beyond the power of existing mathematics. Some general principles throwing light on the question may however be laid down, and in many cases of interest an approximate solution, sufficient for practical purposes, may be obtained.

We know (§§ 79, 242) that the energy of a fluid flowing through a channel cannot be greater than that of any fictitious motion giving the same total current. Hence, if the channel be narrowed in any way, or any obstruction be introduced, the conductivity is thereby diminished, because the alteration is of the nature of an additional constraint. Before the change the fluid was free to adopt the distribution of flow finally assumed. In cases where a rigorous solution cannot be obtained we may use the minimum property to estimate an inferior limit to the conductivity; the energy calculated from a hypothetical law of flow can never be less than the truth, and must exceed it unless the hypothetical and the actual motion coincide.

Another general principle, which is of frequent use, may be more conveniently stated in electrical language. The quantity with which we are concerned is the conductivity of a certain conductor composed of matter of unit specific conductivity. The principle is that if the conductivity of any part of the conductor be increased that of the whole is increased, and if the conductivity of any part be diminished that of the whole is diminished, exception being made of certain very particular cases, where no alteration ensues. In its passage through a conductor electricity distributes itself, so that the energy dissipated is for a given total current the least possible (§ 82). If now the specific resistance of any part be diminished, the total dissipation would be less than before, even if the distribution of currents remained unchanged. *A fortiori* will this be the case, when the currents redistribute themselves so as to make the dissipation a minimum. If an infinitely

thin lamina of matter stretching across the channel be made perfectly conducting, the resistance of the whole will be diminished, unless the lamina coincide with one of the undisturbed equipotential surfaces. In the excepted case no effect will be produced.

306. Among different kinds of channels an important place must be assigned to those consisting of simple apertures in unlimited plane walls of infinitesimal thickness. In practical applications it is sufficient that a wall be very thin in proportion to the dimensions of the aperture, and approximately plane within a distance from the aperture large in proportion to the same quantity.

On account of the symmetry on the two sides of the wall, the motion of the fluid in the plane of the aperture must be normal, and therefore the velocity-potential must be constant; over the remainder of the plane the motion must be exclusively tangential, so that to determine ϕ on one side of the plane we have the conditions (α) ϕ = constant over the aperture, (β) $\dfrac{d\phi}{dn} = 0$ over the rest of the plane of the wall, (γ) ϕ = constant at infinity.

Since we are concerned only with the differences of ϕ we may suppose that at infinity ϕ vanishes. It will be seen that conditions (β) and (γ) are satisfied by supposing ϕ to be the potential of attracting matter distributed over the aperture; the remainder of the problem consists in determining the distribution of matter so that its potential may be constant over the same area. The problem is mathematically the same as that of determining the distribution of electricity on a charged conducting plate situated in an open space, whose form is that of the aperture under consideration, and the conductivity of the aperture may be expressed in terms of the *capacity* of the plate of the statical problem. If ϕ denote the constant potential in the aperture, the electrical resistance (for one side only) will be

$$\phi_1 \div \iint \frac{d\phi}{dn}\, d\sigma,$$

the integration extending over the area of the opening.

Now $\displaystyle\iint \frac{d\phi}{dn}\, d\sigma = 2\pi \times$ (whole quantity of matter distributed),

and thus, if M be the capacity, or charge corresponding to unit-potential, the total resistance is $(\pi M)^{-1}$. Accordingly for the con-

ductivity, which is the reciprocal of the resistance,

$$c = \pi M \text{ ....(1).}$$

So far as I am aware, the ellipse is the only form of aperture for which c or M can be determined theoretically[1], in which case the result is included in the known solution of the problem of determining the distribution of charge on an ellipsoidal conductor. From the fact that a shell bounded by two concentric, similar and similarly situated ellipsoids exerts no force on an internal particle, it is easy to see that the superficial density at any point of an ellipsoid necessary to give a constant potential is proportional to the perpendicular (p) let fall from the centre upon the tangent plane at the point in question. Thus if ρ be the density, $\rho = \kappa p$; the whole quantity of matter Q is given by

$$Q = \iint \rho \, dS = \kappa \iint p \, dS = 4\pi\kappa abc \text{ ....(2),}^{2}$$

so that

$$\rho = \frac{Qp}{4\pi abc} \text{ ....(3).}$$

In the usual notation

$$p = 1 \div \sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}},$$

or, since

$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2},$$

$$p = c \div \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{c^2x^2}{a^4} + \frac{c^2y^2}{b^4}}.$$

If we now suppose that c is infinitely small, we obtain the particular case of an elliptic plate, and if we no longer distinguish between the two surfaces, we get

$$\rho = \frac{Q}{2\pi ab} \div \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \text{ ....(4).}$$

We have next to find the value of the constant potential (P). By considering the value of P at the centre of the plate, we see that

$$P = \iint \frac{\rho \, dS}{r} = \iint \rho \, dr \, d\theta.$$

[1] The case of a resonator with an elliptic aperture was considered by Helmholtz (Crelle, Bd. 57, 1860), whose result is equivalent to (8).

[2] $2c$ being for the moment the third principal axis of the ellipsoid.

Integrating first with respect to r, we have

$$\int_0^r \rho\, dr = Q \div 4a\sqrt{(1-e^2\cos^2\theta)},$$

e being the eccentricity; and thus

$$P = \frac{Q}{a}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-e^2\cos^2\theta)}} = \frac{Q}{a} F(e),$$

where F is the symbol of the complete elliptic function of the first order. Putting $P=1$, we find

$$\frac{c}{\pi} = M = \frac{a}{F(e)} \quad\text{.........................}(5),$$

as the final expression for the capacity of an ellipse, whose semi-major axis is a and eccentricity is e. In the particular case of the circle, $e=0$, $F(e)=\frac{1}{2}\pi$, and thus for a circle of radius R,

$$c = 2R \quad\text{.................................}(6).$$

If the capacity of the resonator be S, we find from (6) § 304

$$\lambda = \pi\sqrt{\left(\frac{2S}{R}\right)} \quad\text{...........................}(7).$$

The area of the ellipse (σ) is given by

$$\sigma = \pi a^2\sqrt{1-e^2},$$

and hence in terms of σ

$$\frac{1}{c} = \frac{1}{2}\sqrt{\left(\frac{\pi}{\sigma}\right)} \cdot \frac{2F(e)(1-e^2)^{\frac{1}{4}}}{\pi} \quad\text{..................}(8).$$

When e is small, we obtain by expanding in powers of e previous to integration,

$$F(e) = \frac{1}{2}\pi\left\{1 + \frac{1^2}{2^2}e^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}e^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}e^6 + \dots\right\} \text{.....}(9),$$

whence

$$\frac{2F(e)(1-e^2)^{\frac{1}{4}}}{\pi} = 1 - \frac{e^4}{64} - \frac{e^6}{64} + \dots$$

Neglecting e^8 and higher powers, we have therefore

$$c = 2\sqrt{\left(\frac{\sigma}{\pi}\right)} \cdot \left(1 + \frac{e^4}{64} + \frac{e^6}{64} + \dots\right) \text{............}(10).$$

From this result we see that, if its eccentricity be small, the conductivity of an elliptic aperture is very nearly the same as that of a circular aperture *of equal area.* Among various forms of aperture of given area there must be one which has a minimum conductivity, and, though a formal proof might be difficult, it is easy to recognise that this can be no other than the circle. An inferior limit to the value of c is thus always afforded by the conductivity of the circle of equal area, that is $2\sqrt{\left(\frac{\sigma}{\pi}\right)}$, and when the true form is nearly circular, this limit may be taken as a close approximation to the real value.

The value of λ is then given by

$$\lambda = 2^{\frac{1}{2}}\pi^{\frac{5}{4}}\sigma^{-\frac{1}{4}}S^{\frac{1}{2}} \quad\text{..........................(11).}$$

In order to shew how slightly a moderate eccentricity affects the value of c, I have calculated the following short table with the aid of Legendre's values of $F(e)$. Putting $e = \sin\psi$, we have $\cos\psi$ as the ratio of axes, and for the conductivity

$$c = 2\sqrt{\left(\frac{\sigma}{\pi}\right)} \cdot \frac{\pi}{2\sqrt{(\cos\psi)} \,.\, F(\sin\psi)}.$$

ψ.	$e = \sin\psi$.	$b : a = \cos\psi$.	$\pi \div 2F(e)(1-e^2)^{\frac{1}{4}}$.
0°	·00000	1·00000	1·0000
20°	·34204	·93969	1·0002
30°	·50000	·86603	1·0013
40°	·64279	·76604	1·0044
50°	·76604	·64279	1·0122
60°	·86603	·50000	1·0301
70°	·93969	·34202	1·0724
80°	·98481	·17365	1·1954
90°	1·00000	·00000	∞

The value of the last factor given in the fourth column is the ratio of the conductivity of the ellipse to *that of a circle of equal area.* It appears that even when the ellipse is so eccentric that the ratio of the axes is 2 : 1, the conductivity is increased by only about 3 per cent., which would correspond to an alteration of little more than a comma (§ 18) in the pitch of a resonator.

There seems no reason to suppose that this approximate independence of shape is a property peculiar to the ellipse, and we may conclude with some confidence that in the case of any moderately elongated oval aperture, the conductivity may be calculated from the area alone with a considerable degree of accuracy.

If the area be given, there is no superior limit to c. For suppose the area σ to be distributed over n equal circles sufficiently far apart to act independently. The area of each circle is $n^{-1}\sigma$, and its conductivity is $2(n\pi)^{-\frac{1}{2}}\sigma^{\frac{1}{2}}$. The whole conductivity is n times as great, and therefore increases indefinitely with n. As a general rule, the more the opening is elongated or broken up, the greater will be the conductivity for a given area.

To find a superior limit to the conductivity of a given aperture we may avail ourselves of the principle that any addition to the aperture must be attended by an increase in the value of c. Thus in the case of a square, we may be sure that c is less than for the circumscribed circle, and we have already seen that it is greater than for the circle of equal area. If b be the side of the square,

$$\frac{2b}{\sqrt{\pi}} < c < \sqrt{2}\, b.$$

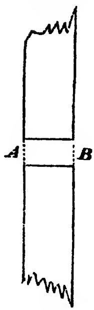
The tones of a resonator with a square aperture calculated from these two limits would differ by about a whole tone; the graver of them would doubtless be much the nearer to the truth. This example shews that even when analysis fails to give a solution in the mathematical sense, we need not be altogether in the dark as to the magnitudes of the quantities with which we are dealing.

In the case of similar orifices, or systems of orifices, c varies as the linear dimension.

307. Most resonators used in practice have necks of greater or less length, and even when there is nothing that would be called a neck, the thickness of the side of the reservoir cannot always be neglected. We shall therefore examine the conductivity of a channel formed by a cylindrical boring through an obstructing plate bounded by parallel planes, and, though we fail to solve the problem rigorously, we shall obtain information sufficient for most practical purposes. The thickness of the plate we shall call L, and the radius of the cylindrical channel R.

Whatever the resistance of the channel may be, it will be lessened by the introduction of infinitely thin discs of perfect conductivity at A and B, fig. 59. The effect of the discs is to produce constant potential over their areas, and the problem thus modified is susceptible of rigorous solution. Outside A and B the motion is the same as that previously investigated, when the obstructing plate is infinitely thin; between A and B the flow is uniform. The resistance is therefore on the whole

Fig. 59.



$$\frac{1}{2R} + \frac{L}{\pi R^2},$$

whence
$$c = \frac{\pi R^2}{L + \frac{1}{2}\pi R} \quad \text{.........................} (1).$$

If α denote the correction, which must be added to L on account of an open end,
$$\alpha = \tfrac{1}{4}\pi R \quad \text{.........................} (2).$$

This correction is in general under the mark, but, when L is very small in comparison with R, the assumed motion coincides more and more nearly with the actual motion, and thus the value of α in (2) tends to become correct.

A superior limit to the resistance may be calculated from a hypothetical motion of the fluid. For this purpose we will suppose infinitely thin pistons introduced at A and B, the effect of which will be to make the normal velocity constant at those places. Within the tube the flow will be uniform as before, but for the external space we have a new problem to consider:—To determine the motion of a fluid bounded by an infinite plane, the normal velocity over a circular area of the plane having a given constant value, and over the remainder of the plane being zero.

The potential may still be regarded as due to matter distributed over the disc, but it is no longer constant over the area; the *density* of the matter, however, being proportional to $\frac{d\phi}{dn}$ is constant.

The kinetic energy of the motion

$$= \tfrac{1}{2}\iint \phi \frac{d\phi}{dn}\, d\sigma = \tfrac{1}{2}\frac{d\phi}{dn}\iint \phi\, d\sigma,$$

the integration going over the area of the circle.

The total current through the plane

$$=\iint \frac{d\phi}{dn}\,d\sigma = \pi R^2 \frac{d\phi}{dn}.$$

Hence $$\frac{2\text{ kinetic energy}}{(\text{current})^2} = \frac{\iint \phi\, d\sigma}{\pi^2 R^4 \frac{d\phi}{dn}}.$$

If the density of the matter be taken as unity, $\frac{d\phi}{dn} = 2\pi$, and the required ratio is expressed by $\frac{P}{\pi^3 R^4}$, where P denotes the potential on itself of a circular layer of matter of unit density and of radius R.

The simplest method of calculating P depends upon the consideration that it represents the work required to break up the disc into infinitesimal elements and to remove them from each other's influence[1]. If we take polar co-ordinates (ρ, θ), the pole being at the edge of the disc whose radius is a, we have for the potential at the pole, $V = \iint d\theta d\rho$, the limits of ρ being 0 and $2a\cos\theta$, and those of θ being $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

Thus $$V = 4a \text{ .... (3).}$$

Now let us cut off a strip of breadth da from the edge of the disc. The work required to remove this to an infinite distance is $2\pi a\,da\,.\,4a$. If we gradually pare the disc down to nothing and carry all the parings to infinity, we find for the total work by integrating with respect to a from 0 to R,

$$P = \frac{8\pi R^3}{3}\;{}^{2}.$$

The limit to the resistance (for one side) is thus $\frac{8}{3\pi^2 R}$; we conclude that the resistance of the whole channel is less than

$$\frac{L}{\pi R^2} + \frac{16}{3\pi^2 R} \text{ .... (4).}$$

Collecting our results, we see that

$$\frac{\pi}{4} R < \alpha < \frac{8}{3\pi} R \text{ .... (5),}$$

[1] A part of § 302 is repeated here for the sake of those who may wish to avoid the difficulties of the more complete investigation.

[2] This method of calculating P was suggested to the author by Professor Clerk Maxwell.

or in decimals,

$$\left.\begin{array}{l} \alpha > \cdot 785\, R \\ \alpha < \cdot 849\, R \end{array}\right\} \qquad (6).$$

It must be observed that α here denotes the correction for one end. The whole resistance corresponds to a length $L + 2\alpha$ of tube having the section πR^2.

When L is very great in relation to R, we may take simply

$$c = \frac{\pi R^2}{L} \qquad (7).$$

In this case we have from (6) § 304

$$\lambda = \frac{2\sqrt{\pi} \,.\, \sqrt{(SL)}}{R} \qquad (8).$$

The correction for an open end (α) is a function of L, coinciding with the lower limit, viz. $\frac{1}{4}\pi R$, when L vanishes. As L increases, α increases with it; but does not, even when L is infinite, attain the superior limit $\frac{8}{3\pi} R$. For consider the motion going on in any middle piece of the tube. The kinetic energy is greater than corresponds merely to the length of the piece. If therefore the piece be removed, and the free ends brought together, the motion otherwise continuing as before, the kinetic energy will be diminished more than corresponds to the length of the piece subtracted. *A fortiori* will this be true of the real motion which would exist in the shortened tube. That, when $L = \infty$, α does not become $\frac{8}{3\pi} R$ is evident, because the normal velocity at the end, far from being constant, as was assumed in the calculation of this result, must increase from the centre outwards and become infinite at the edge.

A further approximation to the value of α may be obtained by assuming a variable velocity at the plane of the mouth. The calculation will be found in Appendix A. It appears that in the case of an infinitely long tube α cannot be so great as $\cdot 82422\, R$. The real value of α is probably not far from $\cdot 82\, R$.

308. Besides the cylinder there are very few forms of channel whose conductivity can be determined mathematically. When however the form is approximately cylindrical we may obtain limits, which are useful as allowing us to estimate the effect

of such departures from mathematical accuracy as must occur in practice.

An inferior limit to the resistance of any elongated and approximately straight conductor may be obtained immediately by the imaginary introduction of an infinite number of plane perfectly conducting layers perpendicular to the axis. If σ denote the area of the section at any point x, the resistance between two layers distant dx will be $\sigma^{-1}dx$, and therefore the whole actual resistance is certainly greater than

$$\int \sigma^{-1} dx \quad\text{.............................. (1),}$$

unless indeed the conductor be truly cylindrical.

In order to find a superior limit we may calculate the kinetic energy of the current on the hypothesis that the velocity parallel to the axis is uniform over each section. The hypothetical motion is that which would follow from the introduction of an infinite number of rigid pistons moving freely, and the calculated result is necessarily in excess of the truth, unless the section be absolutely constant. We shall suppose for the sake of simplicity that the channel is symmetrical about an axis, in which case of course the motion of the fluid is symmetrical also.

If U denote the total current, we have *ex hypothesi* for the axial velocity at any point x

$$u = \sigma^{-1} U \quad\text{.............................. (2),}$$

from which the radial velocity v is determined by the equation of continuity (6 § 238),

$$\frac{d(ru)}{dx} + \frac{d(rv)}{dr} = 0.$$

Thus
$$rv = \text{const.} - \tfrac{1}{2} U r^2 \frac{d\sigma^{-1}}{dx},$$

or, since there is no source of fluid on the axis,

$$v = -\tfrac{1}{2} U r \frac{d\sigma^{-1}}{dx} \quad\text{.............................. (3).}$$

The kinetic energy may now be calculated by simple integration:—

$$\int u^2\sigma\,dx = U^2\int\sigma^{-1}dx,$$

$$\iint v^2 2\pi r\,dr\,dx = \frac{\pi U^2}{8}\int y^4\left(\frac{d\sigma^{-1}}{dx}\right)^2 dx,$$

if y be the radius of the channel at the point x, so that $\sigma = \pi y^2$.

Thus $$\frac{2\text{ kinetic energy}}{(\text{current})^2} = \int\frac{1}{\pi y^2}\left\{1 + \tfrac{1}{2}\left(\frac{dy}{dx}\right)^2\right\}dx \quad\text{.........(4).}$$

This is the quantity which gives a superior limit to the resistance. The first term, which corresponds to the component velocity u, is the same as that previously obtained for the lower limit, as might have been foreseen. The difference between the two, which gives the utmost error involved in taking either of them as the true value, is

$$\frac{1}{2\pi}\int\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 dx \quad\text{........................ (5).}$$

In a nearly cylindrical channel $\dfrac{dy}{dx}$ is a small quantity and so the result found in this manner is closely approximate. It is not necessary that the section should be nearly constant, but only that it should vary slowly. The success of the approximation in this and similar cases depends upon the fact that the quantity to be estimated is at a minimum. Any reasonable approximation to the real motion will give a result very near the truth according to the principles of the differential calculus.

By means of the properties of the potential and stream functions the present problem admits of actual approximate solution. If ϕ and ψ denote the values of these functions at any point x, r; u, v denote the axial and transverse velocities,

$$u = \frac{d\phi}{dx} = \frac{1}{r}\frac{d\psi}{dr}, \qquad v = \frac{d\phi}{dr} = -\frac{1}{r}\frac{d\psi}{dx} \quad\text{............(6),}$$

whence by elimination

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \frac{d^2\phi}{dx^2} = 0 \quad\text{.....................(7),}$$

$$\frac{d^2\psi}{dr^2} - \frac{1}{r}\frac{d\psi}{dr} + \frac{d^2\psi}{dx^2} = 0 \quad\text{..................... (8).}$$

If F denote the value of ϕ as a function of x, when $r=0$, the general values of ϕ and ψ may be expressed in terms of F by means of (7) and (8) in the series

$$\left.\begin{aligned}\phi &= F-\frac{r^2F''}{2^2}+\frac{r^4F^{\text{iv}}}{2^2.4^2}-\frac{r^6F^{\text{vi}}}{2^2.4^2.6^2}+\ldots\\ \psi &= \frac{r^2F'}{2}-\frac{r^4F'''}{2^2.4}+\frac{r^6F^{\text{v}}}{2^2.4^2.6}-\frac{r^8F^{\text{vii}}}{2^2.4^2.6^2.8}+\ldots\end{aligned}\right\}\ldots\ldots\ldots(9),$$

where accents denote differentiation with respect to x. At the boundary of the channel where $r=y$, ψ is constant, say ψ_1. Then

$$\psi_1=\frac{y^2F'}{2}-\frac{y^4F'''}{2^2.4}+\frac{y^6F^{\text{v}}}{2^2.4^2.6}-\ldots\ldots\ldots\ldots(10)$$

is the equation connecting y and F. In the present problem y is given, and we have to express F by means of it. By successive approximation we obtain from (10)

$$F'=\frac{2\psi_1}{y^2}+\frac{y^2}{8}\left\{\frac{d^2}{dx^2}\left(\frac{2\psi_1}{y^2}\right)+\frac{1}{8}\frac{d^2}{dx^2}y^2\frac{d^2}{dx^2}\left(\frac{2\psi_1}{y^2}\right)\right\}$$
$$-\frac{y^4}{12.4^2}\frac{d^4}{dx^4}\left(\frac{2\psi_1}{y^2}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

The total stream is given by the integral

$$\int_0^y\frac{d\phi}{dx}2\pi r\,dr=\int_0^y\frac{1}{r}\frac{d\psi}{dr}2\pi r\,dr=2\pi\psi_1;$$

and therefore the resistance between any two equipotential surfaces is represented by

$$\frac{1}{2\pi\psi_1}\int F'dx.$$

The expression for the resistance admits of considerable simplification by integration by parts in the case when the channel is truly cylindrical in the neighbourhood of the limits of integration. In this way we find for the final result,

$$\text{resistance}=\int\frac{dx}{\pi y^2}\left\{1+\tfrac{1}{2}y'^2-\frac{(3y'^2-yy'')^2}{48}\right\}\ldots\ldots\ldots(12)^1,$$

y', y'' denoting the differential coefficients of y with respect to x.

It thus appears that the superior limit of the preceding investigation is in fact the correct result to the second order of

[1] *Proceedings of the London Mathematical Society*, Vol. VII. No. 93.

approximation. If we regard y as a function of ωx, where ω is a small quantity, (12) is correct as far as terms containing ω^4.

309. Our knowledge of the laws on which the pitch of resonators depends, is due to the labours of several experimenters and mathematicians.

The observation that for a given mouthpiece the pitch of a resonator depends mainly upon the volume S is due to Liscovius, who found that the pitch of a flask partly filled with water was not altered when the flask was inclined. This result was confirmed by Sondhauss[1]. The latter observer found further, that in the case of resonators without necks, the influence of the aperture depended mainly upon its area, although when the shape was very elongated, a certain rise of pitch ensued. He gave the formula

$$N = 52400 \frac{\sigma^{\frac{1}{4}}}{S^{\frac{1}{2}}} \qquad \text{.....................(1)},$$

the unit of length being the millimetre.

The *theory* of this kind of resonator we owe to Helmholtz[2], whose formula is

$$N = \frac{a\,\sigma^{\frac{1}{4}}}{2^{\frac{1}{2}}\,\pi^{\frac{3}{4}}\,S^{\frac{1}{2}}} \qquad \text{.........................(2)},$$

applicable to circular apertures.

For flasks with long necks, Sondhauss[3] found

$$N = 46705 \frac{\sigma^{\frac{1}{2}}}{L^{\frac{1}{2}}\,S^{\frac{1}{2}}} \qquad \text{.........................(3)},$$

corresponding to the theoretical

$$N = \frac{a}{2\pi}\,\frac{\sigma^{\frac{1}{2}}}{L^{\frac{1}{2}}\,S^{\frac{1}{2}}} \qquad \text{.........................(4)}.$$

In practice it does not often happen either that the neck is so long that the correction for the open ends can be neglected, as (4) supposes, or, on the other hand, so short that it can itself be neglected, as supposed in (2). Wertheim[4] was the first

[1] Ueber den Brummkreisel und das Schwingungsgesetz der cubischen Pfeifen. *Pogg. Ann.* LXXXI.

[2] Crelle, Bd. LVII. 1—72. 1860.

[3] Ueber die Schallschwingungen der Luft in erhitzten Glasröhren und in gedeckten Pfeifen von ungleicher Weite. *Pogg. Ann.* LXXIX. 1850.

[4] Mémoire sur les vibrations sonores de l'air. *Ann. d. Chim.* (3) XXXI.

to shew that the effect of an open end could be represented by an addition (α) to the length, independent, or nearly so, of L and λ.

The approximate theoretical determination of α is due to Helmholtz, who gave $\frac{1}{4}\pi R$ as the correction for an open end fitted with an infinite flange. His method consisted in inventing forms of tube for which the problem was soluble, and selecting that one which agreed most nearly with a cylinder. The correction $\frac{1}{4}\pi R$ is rigorously applicable to a tube whose radius at the open end and at a great distance from it is R, but which in the neighbourhood of the open end bulges slightly.

From the fact that the true cylinder may be derived by introducing an obstruction, we may infer that the result thus obtained is too small.

It is curious that the process followed in this work, which was first given in the memoir on resonance, leads to exactly the same result, though it would be difficult to conceive two methods more unlike each other.

The correction to the length will depend to some extent upon whether the flow of air from the open end is obstructed, or not. When the neck projects into open space, there will be less obstruction than when a backward flow is prevented by a flange as supposed in our approximate calculations. However, the uncertainty introduced in this way is not very important, and we may generally take $\alpha = \frac{1}{4}\pi R$ as a sufficient approximation. In practice, when the necks are short, the hypothesis of the flange agrees pretty well with fact, and when the necks are long, the correction is itself of subordinate importance.

The general formula will then run

$$N = \frac{a}{2\pi}\sqrt{\frac{\sigma}{S\,(L + \frac{1}{2}\sqrt{\pi\sigma})}} \quad \text{......................}(5),$$

where σ is the area of the section of the neck, or in numbers

$$N = \frac{a}{6{\cdot}2832}\,\frac{\sigma^{\frac{1}{2}}}{S^{\frac{1}{2}}\sqrt{(L + {\cdot}8863\sqrt{\sigma})}} \quad \text{..................}(6).$$

A formula not differing much from this was given, as the embodiment of the results of his measurements, by Sondhauss[1] who

[1] *Pogg. Ann.* CXL. 53, 219. 1870.

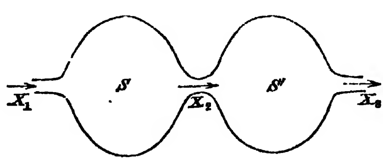
at the same time expressed a conviction that it was no mere empirical formula of interpolation, but the expression of a natural law. The theory of resonators with necks was given about the same time[1] in a memoir 'on Resonance' published in the *Philosophical Transactions* for 1871, from which most of the last few pages is derived.

310. The simple method of calculating the pitch of resonators with which we have been occupied is applicable to the gravest mode of vibration only, the character of which is quite distinct. The overtones of resonators with contracted necks are relatively very high, and the corresponding modes of vibration are by no means independent of the inertia of the air in the interior of the reservoir. The character of these modes will be more evident, when we come to consider the vibrations of air within a completely closed vessel, such as a sphere, but it will rarely happen that the pitch can be calculated theoretically.

There are, however, cases of multiple resonance to which our theory is applicable. These occur when two or more vessels communicate by channels with each other and with the external air; and are readily treated by Lagrange's method, provided of course that the wave-length of the vibration is sufficiently large in comparison with the dimensions of the vessels.

Suppose that there are two reservoirs, S, S', communicating with each other and with the external air by narrow passages or

Fig. 60.



necks. If we were to consider SS' as a single reservoir and apply our previous formula, we should be led to an erroneous result; for that formula is founded on the assumption that within the reservoir the inertia of the air may be left out of account, whereas it is evident that the energy of the motion through the connecting passage may be as great as through the two others. However, an

[1] *Proceedings of the Royal Society*, Nov. 24, 1870.

investigation on the same general plan as before meets the case perfectly. Denoting by X_1, X_2, X_3 the total transfers of fluid through the three passages, we have as in (2) § 304 for the kinetic energy the expression

$$T = \tfrac{1}{2}\rho\left\{\frac{\dot{X}_1^2}{c_1} + \frac{\dot{X}_2^2}{c_2} + \frac{\dot{X}_3^2}{c_3}\right\} \qquad \text{..................(1)},$$

and for the potential energy,

$$V = \tfrac{1}{2}\rho a^2\left\{\frac{(X_2 - X_1)^2}{S} + \frac{(X_3 - X_2)^2}{S'}\right\} \qquad \text{...........(2)}.$$

An application of Lagrange's method gives as the differential equations of motion,

$$\left.\begin{aligned} \frac{\ddot{X}_1}{c_1} + a^2\frac{X_1 - X_2}{S} &= 0 \\ \frac{\ddot{X}_2}{c_2} + a^2\left\{\frac{X_2 - X_1}{S} + \frac{X_2 - X_3}{S'}\right\} &= 0 \\ \frac{\ddot{X}_3}{c_3} + a^2\frac{X_3 - X_2}{S'} &= 0 \end{aligned}\right\} \qquad \text{..............(3)}.$$

By addition and integration,

$$\frac{X_1}{c_1} + \frac{X_2}{c_2} + \frac{X_3}{c_3} = 0 \qquad \text{......................(4)}.$$

Hence on elimination of X_2,

$$\left.\begin{aligned} \ddot{X}_1 + \frac{a^2}{S}\left\{(c_1 + c_2)\, X_1 + \frac{c_1 c_2}{c_3}\, X_3\right\} &= 0 \\ \ddot{X}_3 + \frac{a^2}{S'}\left\{(c_3 + c_2)\, X_3 + \frac{c_3 c_2}{c_1}\, X_1\right\} &= 0 \end{aligned}\right\} \qquad \text{.........(5)}.$$

Assuming $X_1 = Ae^{pt}$, $X_3 = Be^{pt}$, we obtain on substitution and determination of $A : B$,

$$p^4 + p^2 a^2\left\{\frac{c_1 + c_2}{S} + \frac{c_3 + c_2}{S'}\right\} + \frac{a^4}{SS'}\left\{c_1 c_3 + c_2 (c_1 + c_3)\right\} = 0 \text{...(6)},$$

as the equation to determine the natural tones. If N be the frequency of vibration, $N^2 = -\dfrac{p^2}{4\pi^2}$, the two values of p^2 being of course real and negative. The formula simplifies considerably if $c_3 = c_1$, $S' = S$; but it will be more instructive to work out this case from the beginning. Let $c_1 = c_3 = mc_2 = mc$.

The differential equations take the form

$$\left.\begin{aligned}\ddot{X}_1 + \frac{a^2c}{S}\{(1+m)X_1 + X_3\} = 0\\ \ddot{X}_3 + \frac{a^2c}{S}\{(1+m)X_3 + X_1\} = 0\end{aligned}\right\}\quad\text{..............(7)},$$

while from (4) $X_2 = -\dfrac{X_1 + X_3}{m}$.

Hence

$$\left.\begin{aligned}(\ddot{X}_1 + \ddot{X}_3) + \frac{a^2c}{S}(m+2)(X_1 + X_3) = 0\\ (\ddot{X}_1 - \ddot{X}_3) + \frac{a^2c}{S}m(X_1 - X_3) = 0\end{aligned}\right\}\quad\text{.........(8)}.$$

The whole motion may be divided into two parts. For the first of these

$$X_1 + X_3 = 0 \quad\text{.........(9)},$$

which requires that $X_2 = 0$. The motion is therefore the same as might take place were the communication between S and S' cut off, and has its frequency given by

$$N^2 = \frac{a^2c_1}{4\pi^2S} = \frac{a^2mc}{4\pi^2S}\quad\text{.....................(10)}.$$

The density of the air is the same in both reservoirs.

For the other component part, $X_1 - X_3 = 0$, so that

$$X_2 = -\frac{2X_1}{m}, \qquad N'^2 = \frac{a^2(m+2)c}{4\pi^2S}\quad\text{............(11)}.$$

The vibrations are thus opposed in phase. The ratio of frequencies is given by $N'^2 : N^2 = m+2 : m$, shewing that the second mode has the shorter period. In this mode of vibration the connecting passage acts in some measure as a second opening to both vessels, and thus raises the pitch. If the passage be contracted, the interval of pitch between the two notes is small.

A particular case of the general formula worthy of notice is obtained by putting $c_3 = 0$, which amounts to suppressing one of the communications with the external air. We thus obtain

$$p^4 + a^2p^2\left(\frac{c_1 + c_2}{S} + \frac{c_2}{S'}\right) + \frac{a^4c_1c_2}{SS'} = 0\quad\text{...............(12)},$$

or, if $$S = S', \quad c_1 = mc_2 = mc,$$

$$p^4 + a^2p^2\frac{(m+2)c}{S} + \frac{a^4mc^2}{S^2} = 0 \quad\quad (13),$$

whence $$N^2 = \frac{a^2c}{8\pi^2S}\{m + 2 \pm \sqrt{(m^2+4)}\} \quad\quad (14).$$

If we further suppose $m = 1$, or $c_2 = c_1$,

$$N^2 = \frac{a^2c}{8\pi^2S}(3 \pm \sqrt{5}).$$

If N' be the frequency for a simple resonator (S, c),

$$N'^2 = \frac{a^2c}{4\pi^2S},$$

and thus $$N_1^2 : N'^2 = \frac{3+\sqrt{5}}{2} = 2\cdot618,$$

$$N'^2 : N_2^2 = \frac{2}{3-\sqrt{5}} = 2\cdot618.$$

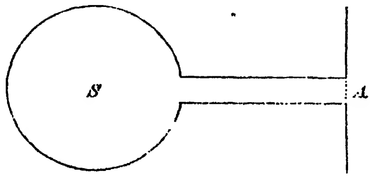
It appears that the interval from N_1 to N' is the same as from N' to N_2, namely, $\sqrt{(2\cdot618)} = 1\cdot618$, or rather more than a fifth. It will be found that whatever the value of m may be, the interval between the two tones cannot be less than $2\cdot414$, which is about an octave and a minor third. The corresponding value of m is 2.

A similar method is applicable to any combination, however complicated, of reservoirs and connecting passages under the single restriction as to the comparative magnitudes of the reservoirs and wave-lengths; but the example just given is sufficient to illustrate the theory of multiple resonance. A few measurements of the pitch of double resonators are detailed in my memoir on resonance, already referred to.

311. The equations which we have employed hitherto take no account of the escape of energy from a resonator. If there were really no transfer of energy between a resonator and the external atmosphere, the motion would be isolated and of little practical interest; nevertheless the characteristic of a resonator consists in its vibrations being in great measure independent. Vibrations, once excited, will continue for a considerable number of periods without much loss of energy, and their frequency will be almost entirely independent of the rate of dissipation. The rate of dissipation is, however, an important feature in the character

of a resonator, on which its behaviour under certain circumstances materially depends. It will be understood that the dissipation here spoken of means only the escape of energy from the vessel and its neighbourhood, and its diffusion in the surrounding medium, and not the transformation of ordinary energy into heat. Of such transformation our equations take no account, unless special terms be introduced for the purpose of representing the effects of viscosity, and of the conduction and radiation of heat.

Fig. 61.



In a previous chapter (§ 278) we saw how to express the motion on the right of the infinite flange (Fig. 61), in terms of the normal velocity of the fluid over the disc A. We found, § 278 (3),

$$\phi = -\frac{1}{2\pi}\iint \frac{d\phi}{dn}\frac{e^{-i\kappa r}}{r}\,d\sigma,$$

where ϕ is proportional to e^{int}.

If r be the distance between any two points of the disc, κr is a small quantity, and $e^{-i\kappa r} = 1 - i\kappa r$ approximately.

Thus $$\phi_A = -\frac{1}{2\pi}\left(\iint \frac{d\phi}{dn}\frac{d\sigma}{r} - i\kappa\iint \frac{d\phi}{dn}\,d\sigma\right) \quad\text{............(1).}$$

The first term depends upon the distribution of the current. If we suppose that $\dfrac{d\phi}{dn}$ is constant, we obtain ultimately a term representing an increase of inertia, or a correction to the length, equal to $\dfrac{8R}{3\pi}$. This we have already considered, under the supposition of a piston at A. The second term, on which the dissipation depends, is independent of the distribution of current,

being a function of the total current ($\dot{X}$) only. Confining our attention to this term, we have

$$\phi_A = \frac{i\kappa \dot{X}}{2\pi} \quad \text{.........................} \quad (2).$$

Assuming now that $\phi \propto e^{int}$, we have for the part of the variation of pressure at A, on which dissipation depends,

$$\delta p = -\rho\,\dot{\phi}_A = -i\rho n \phi_A = \frac{\rho n\kappa \dot{X}}{2\pi} = \frac{\rho n^2 \dot{X}}{2\pi a} \quad \text{..........} (3).$$

The corresponding work done during a transfer of fluid δX is $\frac{\rho n^2 \dot{X}}{2\pi a}\,\delta X$; and since, as in § 304, the expressions for the potential and kinetic energies are

$$V = \tfrac{1}{2}\rho a^2 \frac{X^2}{S}, \qquad T = \tfrac{1}{2}\rho\,\frac{\dot{X}^2}{c} \quad \text{.................} (4),$$

the equation of motion (§ 80) is

$$\ddot{X} + \frac{n^2 c}{2\pi a}\dot{X} + \frac{a^2 c}{S} X = 0 \quad \text{..................} \quad (5),$$[1]

in place of (3) § 304. In the valuation of c an allowance must be included for the inertia of the fluid on the right-hand side of A, corresponding to the term omitted in the expression for δp.

Equation (5) is of the standard form for the free vibrations of dissipative systems of one degree of freedom (§ 45). The amplitude varies as $e^{-\frac{n^2 ct}{4\pi a}}$, being diminished in the ratio $e : 1$ after a time equal to $\frac{4\pi a}{n^2 c}$. If the pitch (determined by n) be given, the vibrations have the greatest persistence when c is smallest, that is, when the neck is most contracted.

If S be given, we have on substituting for c its value in terms of S and n,

$$\frac{4\pi a}{n^2 c} = \frac{4\pi a^3}{n^4 S} \quad \text{........................} (6),$$

shewing that under these circumstances the duration of the motion increases rapidly as n diminishes.

In the case of similar resonators $c \propto n^{-1}$, and then

$$\frac{4\pi a}{n^2 c} \propto \frac{1}{n},$$

[1] Equation (5) is only approximate, inasmuch as the dissipative force is calculated on the supposition that the vibration is permanent; but this will lead to no material error when the dissipation is small.

which shews that in this case the same proportional loss of amplitude always occurs after the lapse of the same number of periods. This result may be obtained by the method of dimensions, as a consequence of the principle of dynamical similarity.

As an example of (5), I may refer to the case of a globe with a neck, intended for burning phosphorus in oxygen gas, whose capacity is ·251 cubic feet. It was found by experiment that the note of maximum resonance made 120 vibrations per second, so that $n = 120 \times 2\pi$. Taking the velocity of sound (a) at 1120 feet per second, we find from these data

$$\frac{4\pi a^3}{n^4 S} = \frac{1}{5} \text{ of a second nearly.}$$

Judging from the sound produced when the globe is struck, I think that this estimate must be too low; but it should be observed that the absence of the infinite flange assumed in the theory must influence very materially the rate of dissipation.

We will now examine the forced vibrations due to a source of sound external to the resonator. If the pressure δp at the mouth of the resonator due to the source, i.e. calculated on the supposition that the mouth is closed, be $Fe^{i\kappa at}$, the equation of motion corresponding to (5), but applicable to the forced vibration only, is

$$\frac{\rho}{c}\ddot{X} + \frac{\rho\kappa^2 a}{2\pi}\dot{X} + \frac{\rho a^2}{S}X = Fe^{i\kappa at} \quad \text{.............(7).}$$

If $X = X_0 e^{i(\kappa at+\epsilon)}$, where X_0 is real,

$$\frac{F^2}{\rho^2 a^4 X_0^2} = \left(\frac{1}{S} - \frac{\kappa^2}{c}\right)^2 + \left(\frac{\kappa^2}{2\pi}\right)^2.$$

The maximum variation of pressure (G) inside the resonator is connected with X_0 by the equation

$$G = \frac{a^2\rho X_0}{S} \quad \text{.......................(8),}$$

since $X_0 \div S$ is the maximum condensation. Thus

$$\frac{F^2}{G^2} = \left(1 - \frac{\kappa^2 S}{c}\right)^2 + \left(\frac{\kappa^2 S}{2\pi}\right)^2 \quad \text{.................(9),}$$

which agrees with the equation obtained by Helmholtz for the case where the communication with the external air is by a simple aperture (§ 306). The present problem is nearly, but not

quite, a case of that treated in § 46, the difference depending upon the fact that the coefficient of dissipation in (7) is itself a function of the period, and not an absolutely constant quantity. If the period, determined by κ, and S be given, (9) shews that the internal variation of pressure (G) is a maximum when $c=\kappa^2 S$, that is, when the natural note of the resonator (calculated without allowance for dissipation) is the same as that of the generating sound. The maximum vibration, when the coincidence of periods is perfect, varies inversely as S; but, if S be small, a very slight inequality in the periods is sufficient to cause a marked falling off in the intensity of the resonance (§ 49). In the practical use of resonators it is not advantageous to carry the reduction of S and c very far, probably because the arrangements necessary for connecting the interior with the ear or other sensitive apparatus involve a departure from the suppositions on which the calculations are founded, which becomes more and more important as the dimensions are reduced. When the sensitive apparatus is not in connection with the interior, as in the experiment of reinforcing the sound of a tuning-fork by means of a resonator, other elements enter into the question, and a distinct investigation is necessary (§ 319).

In virtue of the principle of reciprocity the investigation of the preceding paragraph may be applied to calculate the effect of a source of sound situated in the interior of a resonator.

312. We now pass on to the further discussion of the problem of the open pipe. We shall suppose that the open end of the pipe is provided with an infinite flange, and that its diameter is small in comparison with the wave-length of the vibration under consideration.

As an introduction to the question, we will further suppose that the mouth of the pipe is fitted with a freely moving piston without thickness and mass. The preceding problems, from which the present differs in reality but little, have already given us reason to think that the presence of the piston will cause no important modification. Within the tube we suppose (§ 255) that the velocity-potential is

$$\phi=(A\cos\kappa x+B\sin\kappa x)\,e^{int} \quad \text{..............(1)},$$

where, as usual, $\kappa=2\pi\lambda^{-1}=na^{-1}$. At the mouth, where $x=0$,

$$\phi_0=Ae^{int}; \qquad \left(\frac{d\phi}{dx}\right)_0=\kappa B\,e^{int} \quad \text{............(2)}.$$

On the right of the piston the relation between ϕ_0 and $\left(\frac{d\phi}{dx}\right)_0$ is by § 302

$$\iint \phi_0 d\sigma : \left(\frac{d\phi}{dx}\right)_0 = i\frac{\pi R^2}{\kappa}\left\{1 - \frac{J_1(2\kappa R)}{\kappa R}\right\} - \frac{\pi}{2\kappa^3} K_1(2\kappa R) \text{ ......(3)},$$

R being the radius of the pipe. From this the solution of the problem may be obtained without any restriction as to the smallness of κR: since, however, it is only when κR is small that the presence of the piston would not materially modify the question, we may as well have the benefit of the simplification at once by taking as in (1) § 311

$$\iint \phi_0 d\sigma : \left(\frac{d\phi}{dx}\right)_0 = \frac{i\pi\kappa R^4}{2} - \frac{8R^3}{3} \text{ ....(4)}.$$

Now, since the piston occupies no space, the values of $\left(\frac{d\phi}{dx}\right)_0$ must be the same on both sides of it, and since there is no mass, the like must be true of the values of $\iint\phi_0 d\sigma$. Thus

$$A\sigma = \kappa B\left\{-\frac{8R^3}{3} + i\frac{\kappa\pi R^4}{2}\right\}$$

or

$$A = B\left\{-\frac{8\kappa R}{3\pi} + i\frac{\kappa^2R^2}{2}\right\} \text{ ....(5)}.$$

Substituting in (1), we find on rejecting the imaginary part, and putting for brevity $B = 1$,

$$\phi = \left\{\sin\kappa x - \frac{8\kappa R}{3\pi}\cos\kappa x\right\}\cos nt - \tfrac{1}{2}\kappa^2R^2\cos\kappa x\sin nt \text{......(6)}.$$

In this expression the term containing $\sin nt$ depends upon the dissipation, and is the same as if there were no piston, while that involving $\frac{8\kappa R}{3\pi}$ represents the effect of the inertia of the external air in the neighbourhood of the mouth. In order to compare with previous results, let α be such that

$$\sin\kappa x - \frac{8\kappa R}{3\pi}\cos\kappa x = \sin\kappa(x - \alpha);$$

then, the squares of small quantities being neglected,

$$\alpha = \frac{8R}{3\pi} \text{ ....(7)},$$

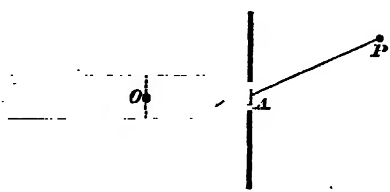
and

$$\phi = \sin\kappa(x-\alpha)\cos nt - \tfrac{1}{2}\kappa^2R^2\cos\kappa x\sin nt \text{............(8)}.$$

These formulæ shew that, if the dissipation be left out of account, the velocity-potential is the same as if the tube were lengthened by $\frac{8}{3\pi}$ of the radius, and the open end then behaved as a loop. The amount of the correction agrees with what previous investigations would have led us to expect as the result of the introduction of the piston. We have seen reason to know that the true value of α lies between $\frac{\pi}{4}R$ and $\frac{8}{3\pi}R$, and that the presence of the piston does not affect the term representing the dissipation. But, before discussing our results, it will be advantageous to investigate them afresh by a rather different method, which besides being of somewhat greater generality, will help to throw light on the mechanics of the question.

313. For this purpose it will be convenient to shift the origin in the negative direction to such a distance from the mouth that the waves are there approximately plane, a displacement which according to our suppositions need not amount to more than a small fraction of the wave-length. The difficulty of the question consists in finding the connection between the waves in the pipe, which at a sufficient distance from the mouth are plane, and the diverging waves outside, which at a moderate distance may be treated as spherical. If the transition take place within a space small compared with the wave-length, which it must evidently do, if the diameter be small enough, the problem admits of solution, whatever may be the form of the pipe in the neighbourhood of the mouth.

Fig. 62.



At a point P, whose distance from A is moderate, the velocity-potential is (§ 279)

$$\psi = \frac{A'}{r} e^{-ikr} e^{int} \qquad (1),$$

whence

$$\frac{d\psi}{dr} = -\frac{A'e^{i(nt-\kappa r)}}{r^2}(1+i\kappa r)\text{.......................}(2).$$

Let us consider the behaviour of the mass of air included between the plane section at O and a hemispherical surface whose centre is A, and radius r, r being large in comparison with the diameter of the pipe, but small in comparison with the wave-length. Within this space the air must move approximately as an incompressible fluid would do. Now the current across the hemispherical surface

$$= 2\pi r^2\frac{d\psi}{dr} = -2\pi A'(1+i\kappa r)e^{i(nt-\kappa r)} = -2\pi A'e^{int}\text{.........}(3),$$

if the square of κr be neglected.

If, as before, we take for the velocity-potential within the pipe

$$\phi = (A\cos\kappa x + B\sin\kappa x)e^{int}\text{....................}(4),$$

we have for the current across the section at O,

$$\sigma\left(\frac{d\phi}{dx}\right)_0 = \sigma\kappa Be^{int}\text{.........................}(5);$$

and thus

$$\sigma\kappa B = -2\pi A'\text{.........................}(6).$$

This is the first condition; the second is to be found from the consideration that the total current (whose two values have just been equated) is proportional to the difference of potential at the terminals. Thus, if c denote the conductivity of the passage between the terminal surfaces,

$$\sigma\left(\frac{d\phi}{dx}\right)_0 = c\,(\psi_r - \phi_0),$$

or

$$\frac{\sigma\kappa B}{c} = \frac{A'}{r}e^{-i\kappa r} - A\text{.......................}(7).$$

On substituting for A' its value from (6), we have

$$-A = \sigma\kappa B\left(\frac{1}{c}+\frac{e^{-i\kappa r}}{2\pi r}\right) = \sigma\kappa B\left\{\frac{1}{c}+\frac{1}{2\pi r}-\frac{i\kappa}{2\pi}\right\}.$$

In this expression the second term is negligible in comparison with the first, for c is at most a quantity of the same order as the radius

of the tube, and when the mouth is much contracted it is smaller still. Thus we may take

$$A = \sigma\kappa B\left(-\frac{1}{c} + \frac{i\kappa}{2\pi}\right) \text{......................(8).}$$

Substituting this in (4), we have for the imaginary expression of the velocity-potential within the tube, if B be put equal to unity,

$$\phi = \left\{\sin\kappa x + \sigma\kappa\left(-\frac{1}{c} + \frac{i\kappa}{2\pi}\right)\cos\kappa x\right\} e^{int},$$

or, if only the real part be retained,

$$\phi = \left\{\sin\kappa x - \frac{\sigma\kappa}{c}\cos\kappa x\right\}\cos nt - \frac{\kappa^2\sigma}{2\pi}\cos\kappa x\sin nt \text{.........(9).}$$

Following Helmholtz, we may simplify our results by introducing a quantity α defined by the equation

$$\tan\kappa\alpha = \frac{\kappa\sigma}{c} \text{..........................(10).}$$

Thus

$$\phi = \frac{\sin\kappa(x-\alpha)}{\cos\kappa\alpha}\cos nt - \frac{\kappa^2\sigma}{2\pi}\cos\kappa x\sin nt \text{............(11),}$$

and the corresponding potential outside the mouth is

$$\psi = -\frac{\sigma\kappa}{2\pi r}\cos(nt - \kappa r) \text{..................(12).}$$

If R be the radius of the tube, we may replace σ by πR^2.

When the tube is a simple cylinder, and the origin lies at a distance ΔL from the mouth, we know that $\sigma c^{-1} = \Delta L + \mu R$, where μ is a number rather greater than $\frac{1}{4}\pi$. In such a case (the origin being taken sufficiently near the mouth) $\kappa\alpha$ is a small quantity, and therefore from (10)

$$\alpha = \frac{\sigma}{c} = \Delta L + \mu R \text{.....................(13).}$$

At the same time $\cos\kappa x$ may be identified with unity. The principal term in ϕ, involving $\cos nt$, may then be calculated, as if the tube were prolonged, and there were a loop at a point situated at a distance μR beyond the actual position of the mouth, in accordance with what we found before. These results, approximate for ordinary tubes, become rigorous when the diameter is reduced without limit, friction being neglected.

If there be no flange at A, the value of c is slightly modified by the removal of what acts as an obstruction, but the principal effect is on the term representing the dissipation. If we suppose as an approximation that the waves diverging from A are spherical, we must take for the current $4\pi r^2 \frac{d\psi}{dr}$ instead of $2\pi r^2 \frac{d\psi}{dr}$. The ultimate effect of the alteration will be to halve the expression for the velocity-potential outside the mouth, as well as the corresponding second term in ϕ (involving $\sin nt$). The amount of dissipation is thus seen to depend materially on the degree in which the waves are free to diverge, and our analytical expressions must not be regarded as more than rough estimates.

The correct theory of the open organ-pipe, including equations (11) and (12), was discovered by Helmholtz[1], whose method, however, differs considerably from that here adopted. The earliest solutions of the problem by Lagrange, D. Bernoulli, and Euler, were founded on the assumption that at an open end the pressure could not vary from that of the surrounding atmosphere, a principle which may perhaps even now be considered applicable to an end whose openness is ideally perfect. The fact that in all ordinary cases energy escapes is a proof that there is not anywhere in the pipe an absolute loop, and it might have been expected that the inertia of the air just outside the mouth would have the effect of an increase in the length. The positions of the nodes in a sounding pipe were investigated experimentally by Savart[2] and Hopkins[3], with the result that the interval between the mouth and the nearest node is always less than the half of that separating consecutive nodes.

314. Experimental determinations of the correction for an open end have generally been made without the use of a flange, and it therefore becomes important to form at any rate a rough estimate of its effect. No theoretical solution of the problem of an unflanged open end has hitherto been given, but it is easy to see that the removal of the flange will reduce the correction materially below the value $\cdot 82\, R$ (Appendix A). In the absence of theory I have attempted to determine the influence of a flange

[1] Crelle, Bd. 57, p. 1. 1860.

[2] Recherches sur les vibrations de l'air. *Ann. d. Chim.* t. xxiv. 1823.

[3] Aerial vibrations in cylindrical tubes. *Cambridge Transactions*, Vol. v. p. 231. 1833.

experimentally[1]. Two organ-pipes nearly enough in unison with one another to give countable beats were blown from an organ bellows; the effect of the flange was deduced from the difference in the frequencies of the beats according as one of the pipes was flanged or not. The correction due to the flange was about $\cdot 2R$. A (probably more trustworthy) repetition of this experiment by Mr Bosanquet gave $\cdot 25R$. If we subtract $\cdot 22R$ from $\cdot 82R$, we obtain $\cdot 6R$, which may be regarded as about the probable value of the correction for an unflanged open end, on the supposition that the wave-length is great in comparison with the diameter of the pipe.

Attempts to determine the correction entirely from experiment have not led hitherto to very precise results. Measurements by Wertheim[2] on doubly open pipes gave as a mean (for each end) $\cdot 663R$, while for pipes open at one end only the mean result was $\cdot 746\,R$. In two careful experiments by Bosanquet[3] on doubly open pipes the correction for one end was $\cdot 635\,R$, when $\lambda = 12\,R$, and $\cdot 543\,R$, when $\lambda = 30\,R$. Bosanquet lays it down as a general rule that the correction (expressed as a fraction of R) increases with the ratio of diameter to wave-length; part of this increase may however be due to the mutual reaction of the ends, which causes the plane of symmetry to behave like a rigid wall. When the pipe is only moderately long in proportion to its diameter, a state of things is approached which may be more nearly represented by the presence than by the absence of a flange. The comparison of theory and observation on this subject is a matter of some difficulty, because when the correction is small, its value, as calculated from observation, is affected by uncertainties as to absolute pitch and the velocity of sound, while for the case, when the correction is relatively larger, which experiment is more competent to deal with, there is at present no theory. Probably a more accurate value of the correction could be obtained from a resonator of the kind considered in § 306, where the communication with the outside air is by a simple aperture; the "length" is in that case zero, and the "correction" is everything. Some measurements of this kind, in which, however, no great accuracy was attempted, will be found in my memoir on resonance[4].

[1] *Phil. Mag.* (5) III. 456. 1877.

[2] *Ann. d. Chim.* (3) t. XXXI. p. 394.

[3] *Phil. Mag.* (5) IV. p. 219. 1877.

[4] *Phil. Trans.* 1871. See also Sondhauss, *Pogg. Ann.* t. 140, 53, 219 (1870), and some remarks thereupon by myself (*Phil. Mag.*, Sept. 1870).

Various methods have been used to determine the pitch of resonators experimentally. Most frequently, perhaps, the resonators have been made to *speak* after the manner of organ-pipes by a stream of air blown obliquely across their mouths. Although good results have been obtained in this way, our ignorance as to the mode of action of the wind renders the method unsatisfactory. In Bosanquet's experiments the pipes were not actually made to speak, but short discontinuous jets of air were blown across the open end, the pitch being estimated from the free vibrations as the sound died away. A method, similar in principle, that I have sometimes employed with advantage consists in exciting free vibrations by means of a blow. In order to obtain as well defined a note as possible, it is of importance to accommodate the hardness of the substance with which the resonator comes into contact to the pitch, a low pitch requiring a soft blow. Thus the pitch of a test-tube may be determined in a moment by striking it against the bent knee.

In using this method we ought not entirely to overlook the fact that the natural pitch of a vibrating body is altered by a term depending upon the square of the dissipation. With the notation of § 45, the frequency is diminished from n to $n(1-\frac{1}{8}\kappa^2 n^{-2})$, or if x be the number of vibrations executed while the amplitude falls in the ratio $e:1$, from n to

$$n\left(1-\frac{1}{8\pi^2 x^2}\right).$$

The correction, however, would rarely be worth taking into account.

The measurements given in my memoir on resonance were conducted upon a different principle by estimating the note of maximum resonance. The ear was placed in communication with the interior of the cavity, while the chromatic scale was sounded. In this way it was found possible with a little practice to estimate the pitch of a good resonator to about a quarter of a semitone. In the case of small flasks with long necks, to which the above method would not be applicable, it was found sufficient merely to hold the flask near the vibrating wires of a pianoforte. The resonant note announced itself by a quivering of the body of the flask, easily perceptible by the fingers. In using this method it is important that the mind should be free from bias in sub-dividing the interval between two consecutive semitones. When the theoretical result

is known, it is almost impossible to arrive at an independent opinion by experiment.

315. We will now, following Helmholtz, examine more closely the nature of the motion within the pipe, represented by the formula (11) § 313. We have

$$\phi = L\cos(nt-\theta) \quad\text{.........................}(1),$$

where
$$L^2 = \frac{\sin^2\kappa(x-\alpha)}{\cos^2\kappa\alpha} + \frac{\kappa^4\sigma^2}{4\pi^2}\cos^2\kappa x \quad\text{...............}(2),$$

$$\tan\theta = -\frac{\kappa^2\sigma_s\cos\kappa\alpha\cos\kappa x}{2\pi\sin\kappa(x-\alpha)} \quad\text{..................}(3).$$

In the expression for L^2 the second term is very small, and therefore the maximum values of ϕ occur very nearly when

$$\kappa(x-\alpha) = (-m+\tfrac{1}{2})\pi,$$

or
$$-x = \tfrac{1}{2}m\lambda - \tfrac{1}{4}\lambda - \alpha \quad\text{........................}(4),$$

where m is a positive integer.

The distance between consecutive maxima is thus $\frac{1}{2}\lambda$, and the value of the maximum is $\sec^2\kappa\alpha$. The minimum values of L^2 occur approximately when $\kappa(x-\alpha) = -m\pi$,

or
$$-x = \tfrac{1}{2}m\lambda - \alpha \quad\text{.........................}(5),$$

and their magnitude is given by

$$L^2 = \frac{\kappa^4\sigma^2}{4\pi^2}\cos^2\kappa x = \frac{\kappa^4\sigma^2}{4\pi^2}\cos^2\kappa\alpha \quad\text{................}(6).$$

In like manner,

$$\frac{d\phi}{dx} = J\cos(nt-\chi) \quad\text{..................}(7),$$

where
$$J^2 = \kappa^2\frac{\cos^2\kappa(x-\alpha)}{\cos^2\kappa\alpha} + \frac{\kappa^6\sigma^2}{4\pi^2}\sin^2\kappa x \quad\text{.............}(8),$$

$$\tan\chi = \frac{\kappa^2\sigma\cos\kappa\alpha\sin\kappa x}{2\pi\cos\kappa(x-\alpha)} \quad\text{.....................}(9).$$

The maximum values of J^2 occur when

$$-x = \tfrac{1}{2}m\lambda - \alpha \quad\text{..........................}(10),$$

and the minimum values, when

$$-x = \tfrac{1}{2}m\lambda - \tfrac{1}{4}\lambda - \alpha \quad\text{.......................}(11).$$

The approximate magnitude of the maximum is $\kappa^2 \sec^2 \kappa x$, and that of the minimum $\kappa^6\sigma^2 \cos^2 \kappa x \div 4\pi^2$. It appears that the maxima of velocity occur in the same parts of the tube as the minima of condensation (and rarefaction), and the minima of velocity in the same places as the maxima of condensation. The series of loops and nodes are arranged as if the first loop were at a distance α beyond the mouth.

With regard to the phases, we see that both θ and χ are in general small; and therefore with the exception of the places where L^2 and J^2 are near their minima the whole motion is synchronous, as if there were no dissipation.

Hitherto we have considered the problem of the passage of plane waves along the pipe and their gradual diffusion from the mouth, without regard to the origin of the plane waves themselves. All that we have assumed is that the origin of the motion is somewhere within the pipe. We will now suppose that the motion is due to the known vibration of a piston, situated at $x = -l$, the origin of co-ordinates being at the mouth. Thus, when $x = -l$,

$$\frac{d\phi}{dx} = G \cos nt \qquad\qquad (12),$$

and this must be made to correspond with the expression for the plane waves, generalized by the introduction of arbitrary amplitude and phase.

We may take

$$\frac{d\phi}{dx} = BJ \cos(nt - \epsilon - \chi) \qquad\qquad (13),$$

where J and χ have the values given in (8), (9), while B and ϵ are arbitrary. Comparing (12) and (13) we conclude that

$$\tan \epsilon = \frac{\kappa^2\sigma \cos\kappa\alpha \sin\kappa l}{2\pi \cos\kappa(l+\alpha)} \qquad\qquad (14),$$

$$G^2 = B^2\kappa^2 \left\{\frac{\cos^2\kappa(l+\alpha)}{\cos^2\kappa\alpha} + \frac{\kappa^4\sigma^2}{4\pi^2}\sin^2\kappa l\right\} \qquad\qquad (15),$$

by which B and ϵ are determined.

In accordance with (12) § 313, the corresponding divergent wave is represented by

$$\psi = -\frac{\sigma\kappa B}{2\pi r}\cos(nt - \epsilon - \kappa r) \qquad\qquad (16).$$

If G be given, B is greatest, when $\cos\kappa(l+\alpha)=0$, that is when the piston is situated at an approximate node. In that case

$$B=\frac{2\pi}{\kappa^3\sigma\cos\kappa\alpha}G\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (17),$$

shewing that the magnitude of the resulting vibration is very great, though not infinite, since $\cos\kappa\alpha$ cannot vanish. When the mouth is much contracted, $\cos\kappa\alpha$ may become small, but in this case it is necessary that the adjustment of periods be very exact in order that the first term of (15) may be negligible in comparison with the second. In ordinary pipes $\cos\kappa\alpha$ is nearly equal to unity.

The minimum of vibration occurs when l is such that $\cos\kappa(l+\alpha)=\pm 1$, that is, when the piston is situated at a loop. In that case

$$B=\frac{G\cos\kappa\alpha}{\kappa}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (18).$$

The vibration outside the tube is then, according to the value of α, equal to or smaller than the vibration which there would be if there were no tube and the vibrating plate were made part of the yz plane.

316. Our equations may also be applied to the investigation of the motion excited in a tube by external sources of sound. Let us suppose in the first place that the mouth of the tube is closed by a fixed plate forming part of the yz plane, and that the potential due to the external sources (approximately constant over the plate) is under these circumstances

$$\psi=H\cos nt\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where ψ is composed of the potential due to each source and its image in the yz plane, as explained in § 278. Inside the tube let the potential be

$$\phi=H\cos\kappa x\cos nt\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

so that ϕ and its differential coefficient are continuous across the barrier. The physical meaning of this is simple. We imagine within the tube such a motion as is determined by the conditions that the velocity at the mouth is zero, and that the condensation at the mouth is the same as that due to the sources of sound when the mouth is closed. It is obvious that under these circumstances

the closing plate may be removed without any alteration in the motion. Now, however, there is in general a finite velocity at $x=-l$, and therefore we cannot suppose the pipe to be there stopped. But when there happens to be a node at $x=-l$, that is to say when l is such that $\cos\kappa(l+\alpha)=0$, all the conditions are satisfied, and the actual motion within the pipe is that expressed by (2). This motion is evidently the same as might obtain, if the pipe were closed at both ends; and in external space the potential is the same as if the mouth of the pipe were closed with the rigid plate.

In the general case in order to reduce the air at $x=-l$ to rest, we must superpose on the motion represented by (2) another of the kind investigated in § 313, so determined as to give at $x=-l$ a velocity equal and opposite to that of the first. Thus, if the second motion be given by

$$\frac{d\phi}{dx} = B\cos(nt-\epsilon-\chi),$$

we have $\epsilon+\chi=0$, and

$$B^2\left\{\frac{\cos^2\kappa(l+\alpha)}{\cos^2\kappa\alpha} + \frac{\kappa^4\sigma^2}{4\pi^2}\sin^2\kappa l\right\} = H^2\sin^2\kappa l \quad\ldots\ldots\ldots(3).$$

When $\sin\kappa l=0$, we have, as above explained, $B=0$. The maximum value of B occurs when $\cos\kappa(l+\alpha)=0$, and then

$$B=\frac{2\pi H}{\kappa^2\sigma}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)^1.$$

It appears, as might have been expected, that the resonance is greatest when the reduced length is an odd multiple of $\frac{1}{4}\lambda$.

317. From the principle that in the neighbourhood of a node the inertia of the air does not come much into play, we see that in such places the form of a tube is of little consequence, and that only the capacity need be attended to. This consideration allows us to calculate the pitch of a pipe which is cylindrical through most of its length (l), but near the closed end expands into a bulb of small capacity (S). The reduced length is then evidently

$$l+\alpha+S\sigma^{-1}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

[1] Helmholtz, *Crelle*, 1860.

where α is the correction for the open end, and σ is the area of the transverse section of the cylindrical part. This formula is often useful, and may be applied also when the deviation from the cylindrical form does not take the shape of an enlargement.

When the enlargement represented by S is too large to allow of the above treatment, we may proceed as follows. The dissipation being neglected, the velocity potential in the tube may be taken to be

$$\phi = \sin \kappa (x - \alpha) \cos nt,$$

the origin being at the mouth, while $\alpha = \frac{1}{4}\pi R$ approximately. At $x = -l$, we have

$$\dot{\phi} = n \sin \kappa (l + \alpha) \sin nt,$$

and

$$\frac{d\phi}{dx} = \kappa \cos \kappa (l + \alpha) \cos nt.$$

Now the condensation is given by $s = -a^{-2}\dot{\phi}$, and the condition to be satisfied at $x = -l$ is

$$S\frac{ds}{dt} = -\sigma \frac{d\phi}{dx} \quad\text{..............................(2),}$$

if it be assumed that the condensation within S is sensibly uniform. Thus

$$Sn^2a^{-2} \sin \kappa (l + \alpha) = \sigma\kappa \cos \kappa (l + \alpha),$$

or, since $n = a\kappa$,

$$\tan \kappa (l + \alpha) = \frac{\sigma}{\kappa S} \quad\text{..........................(3)}$$

is the equation determining the pitch. Numerical examples of the application of (3) are given in my memoir on resonance (*Phil. Trans.* 1871, p. 117).

Similar reasoning proves that in any case of stationary vibrations, for which the wave-length is several times as great as the diameter of the bulb, the end of the tube adjoining the bulb behaves approximately as an open end if κS be much greater than σ, and as a stopped end if κS be much less than σ.

318. The action of a resonator when under the influence of a source of sound in unison with itself is a point of considerable delicacy and importance, and one on which there has been a

good deal of confusion among acoustical writers, the author not excepted.

There are cases where a resonator absorbs sound, as it were attracting the vibrations to itself and so diverting them from regions where otherwise they would be felt. For example, suppose that there is a simple source of sound B situated in a narrow tube at a distance $\frac{1}{4}\lambda$ (or any odd multiple thereof) from a closed end, and not too near the mouth: then at any distant external point A, its effect is nil. This is an immediate consequence of the principle of reciprocity, because if A were the source, there could be no variation of potential at B. The restriction, precluding too great a proximity to the mouth, may be dispensed with, if we suppose the source B to be diffused uniformly over the cross section, instead of concentrated in one point. Then, whatever may be the size and shape of the section, there is absolutely no disturbance on the further side. This is clear from the theory of vibrations in one dimension; the reciprocal form of the proposition—that whatever sources of disturbance may exist beyond the section, $\iint \psi d\sigma = 0$—may be proved from Helmholtz's formula (2) § 293, by taking for ϕ the velocity potential of the purely axial vibration of the same period.

It is scarcely necessary to say that, whenever no energy is emitted, the source does no work; and this requires, not that there shall be no variation of pressure at the source, for that in the case of a simple source is impossible, but that the variable part of the pressure shall have exactly the phase of the acceleration, and no component with the phase of the velocity.

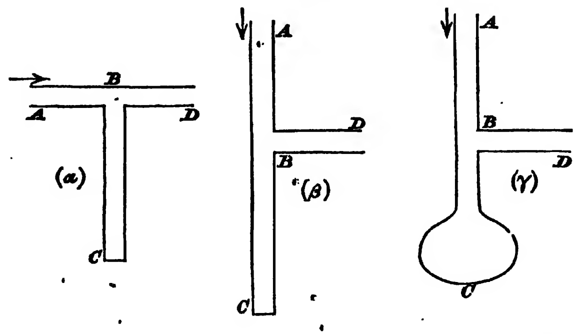
Other examples of the absorption of sound by resonators are afforded by certain modifications of Herschel's interference tube used by Quincke[1] to stop tones of definite pitch from reaching the ear.

In the combinations of pipes represented in Fig. 63, the sound enters freely at A; at B it finds itself at the mouth of a resonator of pitch identical with its own. Under these circumstances it is absorbed, and there is no vibration propagated along BD. It is clear that the cylindrical tube BC may be replaced by any other resonator of the same pitch (γ), without prejudice to the

[1] Pogg. *Ann.* CXXVIII. 177. 1866.

action of the apparatus. The ordinary explanation by interference (so called) of direct and reflected waves is then less applicable.

Fig. 63.



These cases where the source is at the mouth of a resonator must not be confused with others where the source is in the interior. If B be a source at the bottom of a stopped tube whose reduced length is $\frac{1}{4}\lambda$, the intensity at an external point A may be vastly greater than if there had been no tube. In fact the potential at A due to the source at B is the same as it would be at B were the source at A.

319. For a closer examination of the mechanics of resonance, we shall obtain the problem in a form disembarrassed of unnecessary difficulties by supposing the resonator to consist of a small circular plate, backed by a spring, and imbedded in an indefinite rigid plane. It was proved in a previous chapter, (30) § 302, that if M be the mass of the plate, ξ its displacement, $\mu\xi$ the force of restitution, R the radius, and σ the density of the air, the equation of vibration is

$$\left(M+\frac{8\sigma R^3}{3}\right)\ddot{\xi}+\frac{a\sigma\pi\kappa^2R^4}{2}\dot{\xi}+\mu\,\xi=F \quad \ldots\ldots\ldots\ldots(1),$$

where F and ξ are proportional to $e^{i\kappa at}$.

If the natural period of vibration (the reaction of external air included) coincide with that imposed, the equation reduces to

$$\tfrac{1}{2}\,a\sigma\pi\kappa^2R^4\,\dot{\xi}=F \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Let us now suppose that F is due to an external source of sound, giving when the plate is at rest a potential ψ_0, which will be nearly constant over the area of the plate. Thus

$$F = -\delta p \,.\, \pi R^2 = i\kappa a \sigma \,.\, \pi R^2 \,.\, \psi_0 \qquad \text{.................. (3)};$$

so that

$$\pi R^2 \dot{\xi} = \dot{X} = 2 i \pi \kappa^{-1} \psi_0 = i\lambda\psi_0 \qquad \text{.................. (4)},$$

and the potential ϕ due to the motion of the plate at a distance r will be

$$\phi = -\frac{\dot{X}}{2\pi}\frac{e^{-i\kappa r}}{r} = -\frac{i\psi_0}{\kappa}\frac{e^{-i\kappa r}}{r} = \psi_0 \frac{e^{-i\kappa r}}{i\kappa r} \qquad \text{.............. (5)},$$

independent, it should be observed, of the area of the plate.

Leaving for the present the case of perfect isochronism, let us suppose that

$$-\left(M + \frac{8\sigma R^3}{3}\right)\kappa'^2 a^2 + \mu = 0 \qquad \text{.................. (6)},$$

so that $2\pi\kappa'^{-1}$ is the wave-length of the natural note of the resonator. If M' be written for $M + \frac{8}{3}\sigma R^3$, the equation corresponding to (5) takes the form

$$\phi = \psi_0 \frac{e^{-i\kappa r}}{i\kappa r} \div \left[1 - 2iM' \frac{\kappa'^2 - \kappa^2}{\pi\sigma\kappa^3 R^4}\right] \qquad \text{.................. (7)},$$

from which we may infer as before that if $\kappa' = \kappa$ the efficiency of the resonator as a source is independent of R. When the adjustment is imperfect, the law of falling off depends upon $M'R^{-4}$. Thus if M' be great and R small, although the maximum efficiency of the resonator is no less, a greater accuracy of adjustment is required in order to approach the maximum (§ 49). In the case of resonators with simple apertures $M' = \frac{16}{3}\sigma R^3$, so that $M'R^{-4}$ varies as R^{-1}. Accordingly resonators with small apertures require the greatest precision of tuning, but the difference is not important. From a comparison of the present investigation with that of § 311 it appears that the conditions of efficiency are different according as internal or external effects are considered.

We will now return to the case of isochronism and suppose further that the external source of sound to which the resonator A responds, is the motion of a similar plate B, whose distance c from A is a quantity large in comparison with the dimensions

of the plates. The intensity of B may be supposed to be such that its potential is

$$\psi = \frac{e^{-ikr}}{r} \qquad\text{.........................(8).}$$

Accordingly $\psi_0 = c^{-1} e^{-ikc}$, and therefore by (5)

$$\phi = \psi_0 \frac{e^{-ikr}}{ikr} = \frac{e^{-ikc}}{ikc} \cdot \frac{e^{-ikr}}{r} \qquad\text{....................(9),}$$

shewing that at equal distances from their sources

$$\phi : \psi = e^{-ikc} : ikc \qquad\text{....................(10).}$$

The relation of phases may be represented by regarding the induced vibration ϕ as proceeding from B by way of A, and as being subject to an additional retardation of $\frac{1}{4}\lambda$, so that the whole retardation between B and A is $c + \frac{1}{4}\lambda$. In respect of amplitude ϕ is greater than ψ in the ratio of $1 : kc$.

Thus when kc is small, the induced vibration is much the greater, and the total sound is much louder than if A were not permitted to operate. In this case the phase is retarded by a quarter of a period.

It is important to have a clear idea of the cause of this augmentation of sound. In a previous chapter (§ 280) we saw that, when A is fixed, B gives out much less sound than might at first have been expected from the pressure developed. The explanation was that the *phase* of the pressure was unfavourable; the larger part of it is concerned only in overcoming the inertia of the surrounding air, and is ineffective towards the performance of work. Now the pressure which sets A in motion is the whole pressure, and not merely the insignificant part that would of itself do work. The motion of A is determined by the condition that that component of the whole pressure upon it, which has the phase of the velocity, shall vanish. But of the pressure that is due to the motion of A, the larger part has the phase of the acceleration; and therefore the prescribed condition requires an equality between the small component of the pressure due to A's motion, and a pressure comparable with the large component of the pressure due to B's motion. The result is that A becomes a much more powerful source than B. Of course no work is done by the piston A; its effect is to augment the work done at B,

by modifying the otherwise unfavourable relation between the phases of the pressure and of the velocity.

The infinite plane in the preceding discussion is only required in order that we may find room behind it for our machinery of springs. If we are content with still more highly idealized sources and resonators, we may dispense with it. To each piston must be added a duplicate, vibrating in a similar manner, but in the opposite direction, the effect of which will be to make the normal velocity of the fluid vanish over the plane AB. Under these circumstances the plane is without influence and may be removed. If the size of the plates be reduced without limit they become ultimately equivalent to simple sources of fluid; and we conclude that a simple source B will become more efficient than before in the ratio of $1 : \kappa c$, when at a small distance c from it there is allowed to operate a simple resonator (as we may call it) of like pitch, that is, a source in which the inertia of the immediately surrounding fluid is compensated by some adequate machinery, and which is set in motion by external causes only.

In the present state of our knowledge of the mechanics of vibrating fluids, while the difficulties of deduction are for the most part still to be overcome, any simplification of conditions which allows progress to be made, without wholly destroying the practical character of the question, may be a step of great importance. Such, for example, was the introduction by Helmholtz of the idea of a source concentrated in one point, represented analytically by the violation at that point of the equation of continuity. Perhaps in like manner the idea of a simple resonator may be useful, although the thing would be still more impossible to construct than a simple source.

320. We have seen that there is a great augmentation of sound, when a suitably tuned resonator is close to a simple source. Much more is this the case, when the source of sound is compound. The potential due to a double source is (§§ 294, 324)

$$r\psi = \mu e^{-i\kappa r}\left(1 + \frac{1}{i\kappa r}\right) \quad \text{.....................(1).}$$

If the resonator be at a small distance c,

$$\psi_0 = \mu_0 \frac{e^{-i\kappa c}}{i\kappa c^2},$$

and therefore the potential due to the resonator at a distance r' is

$$\phi = \mu_0 \frac{e^{-i\kappa c}}{i\kappa c^2} \cdot \frac{e^{-i\kappa r'}}{i\kappa r'} = \mu_0 \frac{e^{-i\kappa c}}{i^2\kappa^2 c^2} \cdot \frac{e^{-i\kappa r'}}{r'} \quad\text{.............. (2).}$$

If μ_0 vanish, the resonator is without effect; but when $\mu_0 = \pm 1$, that is, when the resonator lies on the axis of the double source, we have

$$\phi = \mp \frac{e^{-i\kappa c}}{\kappa^2 c^2} \cdot \frac{e^{-i\kappa r'}}{r'} \quad\text{...................... (3).}$$

At a distance from the double source its potential is

$$\psi = \mu \frac{e^{-i\kappa r}}{r} \quad\text{.......................... (4).}$$

Thus we may consider that the potential due to the resonator is greater than that due to the double source in the ratio $\kappa^2 c^2 : 1$, the angular variation being disregarded.

A vibrating rigid sphere gives the same kind of motion to the surrounding air as a double source situated at its centre; but the substitution suggested by this fact is only permissible when the radius of the sphere is small in comparison with c: otherwise the presence of the sphere modifies the action of the resonator. Nevertheless the preceding investigation shews how powerful in general the action of a resonator is when placed in a suitable position close to a compound source of sound, whose character is such that it would of itself produce but little effect at a distance.

One of the best examples of this use of a resonator is afforded by a vibrating bar of glass, or metal, held at the nodes. A strip of plate glass about a foot long and an inch broad, of medium thickness (say $\frac{1}{8}$ inch), supported at about 3 inches from the ends by means of string twisted round it, answers the purpose very well. When struck by a hammer it gives but little sound except overtones; and even these may almost be got rid of by choosing a hammer of suitable softness. This deficiency of sound is a consequence of the small dimensions of the bar in comparison with the wave-length, which allows of the easy transference of air from one side to the other. If now the mouth of a resonator of the right pitch[1] be held over one of the free ends, a sound of con-

[1] To get the best effect, the mouth of the resonator ought to be pretty close to the bar; and then the pitch is decidedly lower than it would be in the open. The final adjustment may be made by varying the amount of obstruction. This use of resonators is of great antiquity.

siderable force and purity may be obtained by a well managed blow. In this way an improved harmonicon may be constructed, with tones much lower than would be practicable without resonators. In the ordinary instrument the wave-lengths are sufficiently short to permit the bar to communicate vibrations to the air independently.

The reinforcement of the sound of a bell in a well-known experiment due to Savart[1] is an example of the same mode of action; but perhaps the most striking instance is in the arrangement adopted by Helmholtz in his experiments requiring pure tones, which are obtained by holding tuning-forks over the mouths of resonators.

321. When two simple resonators A_1, A_2, separately in tune with the source, are close together, the effect is less than if there were only one. If the potentials due respectively to A_1, A_2 be ϕ_1, ϕ_2, we may take

$$\phi_1 = A_1 \frac{e^{-ikr_1}}{r_1}, \qquad \phi_2 = A_2 \frac{e^{-ikr_2}}{r_2}.$$

Let R represent the distance A_1A_2, and ψ_1, ψ_2, the potentials that would exist at A_1, A_2, if there were no resonators; then the conditions to determine A_1, A_2 are by (5) § 319

$$\left.\begin{aligned} \psi_1 + \frac{A_2}{R} &= +ik A_1 \\ \psi_2 + \frac{A_1}{R} &= +ik A_2 \end{aligned}\right\} \qquad (1).$$

By hypothesis ψ_1 and ψ_2 are nearly equal, and therefore

$$A_1 = A_2 = \frac{R}{-1 + ikR}\,\psi \qquad (2).$$

Since ikR is small, the effect is much less than if there were only one resonator. It must be observed however that the diminished effectiveness is due to the resonators putting one another out of tune, and if this tendency be compensated by an alteration in the spring, any number of resonators near together have just the effect of one. This point is illustrated by § 302, where it will be seen (32) that though the resonance does not depend upon the size of the plate, still the inertia of the air, which has to be compensated by a spring, does depend upon it.

[1] *Ann. d. Chim.* t. XXIV. 1823.

322. It will be proper to say a few words in this place on an objection, which has been brought forward by Bosanquet[1] as possibly invalidating the usual calculations of the pitch of resonators and of the correction to the length of organ pipes. When fluid flows in a steady stream through a hole in a thin plate, the motion on the low pressure side is by no means of the character investigated in § 306. Instead of diverging after passing the hole so as to follow the surface of the plate, the fluid shapes itself into an approximately cylindrical jet, whose form for the case of two dimensions can be calculated from formulæ given by Kirchhoff[2]. On the high pressure side the motion does not deviate so widely from that determined by the electrical law. In like manner fluid passing outwards from a pipe continues to move in a cylindrical stream. If the external pressure be the greater, the character of the motion is different. In this case the stream lines converge from all directions to the mouth of the pipe, afterwards gathering themselves into a parallel bundle, whose section is considerably less than that of the pipe. It is clear that, if the formation of jets took place to any considerable extent during the passage of air through the mouths of resonators, our calculations of pitch would have to be seriously modified.

The precise conditions under which jets are formed is a subject of great delicacy. It may even be doubted whether they would occur at all in frictionless fluid moving with velocities so small that the corresponding pressures, which are proportional to the squares of the velocities, are inconsiderable. But with air, as we actually have it, moving under the action of the pressures to be found in resonators, it must be admitted that jets may sometimes occur. While experimenting about two years ago with one of König's brass resonators of pitch c', I noticed that when the corresponding fork, strongly excited, was held to the mouth, a wind of considerable force issued from the nipple at the opposite side. This effect may rise to such intensity as to blow out a candle upon whose wick the stream is directed. It does not depend upon any peculiar motion of the air near the ends of the fork, as is proved by mounting the fork upon its resonance-box and presenting the open end of the box, instead of the fork itself, to the mouth of the resonator, when the effect is obtained with but slightly diminished

[1] *Phil. Mag.* Aug. 1877, p. 125.
[2] *Phil. Mag.* Dec. 1876.

intensity. A similar result was obtained with a fork and resonator, of pitch an octave lower (c). Closer examination revealed the fact that at the sides of the nipple the outward flowing stream was replaced by one in the opposite direction, so that a tongue of flame from a suitably placed candle appeared to enter the nipple at the same time that another candle situated immediately in front was blown away. The two effects are of course in reality alternating, and only appear to be simultaneous in consequence of the inability of the eye to follow such rapid changes. The formation of jets must make a serious draft on the energy of the motion, and this is no doubt the reason why it is necessary to close the nipple in order to obtain a powerful sound from a resonator of this form, when a suitably tuned fork is presented to it.

At the same time it does not appear probable that jet formation occurs to any appreciable extent at the mouths of resonators as ordinarily used. The near agreement between the observed and the calculated pitch is almost a sufficient proof of this. Another argument tending to the same conclusion may be drawn from the persistence of the free vibrations of resonators (§ 311), whose duration seems to exclude any important cause of dissipation beyond the communication of motion to the surrounding air.

In the case of organ pipes, where the vibrations are very powerful, these arguments are less cogent, but I see no reason for thinking that the motion at the upper open end differs greatly from that supposed in Helmholtz's calculation. No conclusion to the contrary can, I think, safely be drawn from the phenomena of steady motion. In the opposite extreme case of impulsive motion jets certainly cannot be formed, as follows from Thomson's principle of least energy (§ 79), and it is doubtful to which extreme the case of periodic motion may with greatest plausibility be assimilated. Observation by the method of intermittent illumination (§ 42) might lead to further information upon this subject.

CHAPTER XVII.

APPLICATIONS OF LAPLACE'S FUNCTIONS.

323. THE general equation of a velocity potential, when referred to polar co-ordinates, takes the form (§ 241)

$$r^2\frac{d^2\psi}{dr^2}+2r\frac{d\psi}{dr}+\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi}{d\theta}\right)+\frac{1}{\sin^2\theta}\frac{d^2\psi}{d\omega^2}+\kappa^2r^2\psi=0\ldots(1).$$

If κ vanish, we have the equation of the ordinary potential, which, as we know, is satisfied, if $\psi=r^nS_n$, where S_n denotes the spherical surface harmonic[1] of order n. On substitution it appears that the equation satisfied by S_n is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS_n}{d\theta}\right)+\frac{1}{\sin^2\theta}\frac{d^2S_n}{d\omega^2}+n(n+1)S_n=0\ldots\ldots(2).$$

Now, whatever the form of ψ may be, it can be expanded in a series of spherical harmonics

$$\psi=\psi_0+\psi_1+\psi_2+\ldots\ldots+\psi_n+\ldots\ldots\ldots\ldots\ldots(3),$$

where ψ_n will satisfy an equation such as (2).

Comparing (1) and (2) we see that to determine ψ_n as a function of r, we have

$$r^2\frac{d^2\psi_n}{dr^2}+2r\frac{d\psi_n}{dr}-n(n+1)\psi_n+\kappa^2r^2\psi_n=0;$$

or, as it may also be written,

$$\frac{d^2(r\psi_n)}{d(\kappa r)^2}-\frac{n(n+1)}{(\kappa r)^2}(r\psi_n)+r\psi_n=0\ldots\ldots\ldots\ldots(4).$$

[1] On the theory of these functions the latest English works are Todhunter's *The Functions of Laplace, Lamé, and Bessel*, and Ferrers' *Spherical Harmonics*.

In order to solve this equation, we may observe that when r is very great, the middle term is relatively negligible, and that then the solution is

$$r\psi_n = Ae^{\iota\kappa r} + Be^{-\iota\kappa r} \quad \text{.....................(5).}$$

The same form may be assumed to hold good for the complete equation (4), if we look upon A and B no longer as constants, but as functions of r, whose nature is to be determined. Substituting in (4), we find for B,

$$-\frac{d^2B}{d(\iota\kappa r)^2} + 2\frac{dB}{d(\iota\kappa r)} + \frac{n(n+1)}{(\iota\kappa r)^2}B = 0 \quad \text{...........(6).}$$

Let us assume

$$B = B_0 + B_1(\iota\kappa r)^{-1} + B_2(\iota\kappa r)^{-2} + \ldots + B_s(\iota\kappa r)^{-s} + \ldots \text{(7)},$$

and substitute in (6). Equating to zero the coefficient of $(\iota\kappa r)^{-s-2}$, we obtain

$$B_{s+1} = B_s\frac{n(n+1) - s(s+1)}{2(s+1)} = B_s\frac{(n-s)(n+s+1)}{2(s+1)} \quad \text{.....(8).}$$

Thus
$$B_1 = \frac{n(n+1)}{2}B_0,$$

$$B_2 = B_1\frac{(n-1)(n+2)}{2.2} = \frac{(n-1)n(n+1)(n+2)}{2.4}B_0, \quad \text{\&c.;}$$

so that

$$B = B_0\left\{1 + \frac{n(n+1)}{2.\iota\kappa r} + \frac{(n-1)\ldots(n+2)}{2.4.(\iota\kappa r)^2} + \frac{(n-2)\ldots(n+3)}{2.4.6.(\iota\kappa r)^3} + \ldots + \frac{1.2.3\ldots 2n}{2.4.6\ldots 2n.(\iota\kappa r)^n}\right\} \quad \text{...............(9).}$$

Denoting with Prof. Stokes[1] the series within brackets by $f_n(\iota\kappa r)$, we have

$$B = B_0 f_n(\iota\kappa r) \quad \text{.......................(10).}$$

In like manner by changing the sign of ι, we get

$$A = A_0 f_n(-\iota\kappa r) \quad \text{.......................(11).}$$

The symbols A_0 and B_0, though independent of r, are functions of the angular co-ordinates: in the most general case, they are any two spherical surface harmonics of order n. Equation (5) may therefore be written

$$r\psi_n = S_n e^{-\iota\kappa r} f_n(\iota\kappa r) + S_n' e^{+\iota\kappa r} f_n(-\iota\kappa r) \quad \text{.........(12).}$$

[1] On the Communication of Vibrations from a Vibrating Body to a surrounding Gas. *Phil. Trans.* 1868.

By differentiation of (12)

$$\frac{d\psi_n}{dr} = -\frac{S_n}{r^2} e^{-i\kappa r} F_n(i\kappa r) - \frac{S_n'}{r^2} e^{+i\kappa r} F_n(-i\kappa r) \quad \text{....(13)},$$

where

$$F_n(i\kappa r) = (1 + i\kappa r) f_n(i\kappa r) - i\kappa r f_n'(-i\kappa r) \quad \text{......(14)}.$$

The forms of the functions F, as far as $n = 7$, are exhibited in the accompanying table:

$F_0(y) = y + 1$
$F_1(y) = y + 2 + 2y^{-1}$
$F_2(y) = y + 4 + 9y^{-1} + 9y^{-2}$
$F_3(y) = y + 7 + 27y^{-1} + 60y^{-2} + 60y^{-3}$
$F_4(y) = y + 11 + 65y^{-1} + 240y^{-2} + 525y^{-3} + 525y^{-4}$
$F_5(y) = y + 16 + 135y^{-1} + 735y^{-2} + 2625y^{-3} + 5670y^{-4} + 5670y^{-5}$
$F_6(y) = y + 22 + 252y^{-1} + 1890y^{-2} + 9765y^{-3} + 34020y^{-4} + 72765y^{-5} + 72765y^{-6}$
$F_7(y) = y + 29 + 434y^{-1} + 4284y^{-2} + 29925y^{-3} + 148995y^{-4} + 509355y^{-5} + 1081080y^{-6}$
$\quad + 1081080y^{-7}$

In order to find the leading terms in $F_n(i\kappa r)$ when $i\kappa r$ is small, we have on reversing the series in (9)

$$f_n(i\kappa r) = 1 \cdot 3 \cdot 5 \ldots (2n-1)(i\kappa r)^{-n}\left\{1 + i\kappa r + \frac{n-1}{2n-1}(i\kappa r)^2 + \ldots\right\} \quad \text{..............................(15)},$$

whence by (14) we find

$$F_n(i\kappa r) = 1 \cdot 3 \cdot 5 \ldots (2n-1)(n+1)(i\kappa r)^{-n} \times \left\{1 + i\kappa r + \frac{n^2 (i\kappa r)^2}{(n+1)(2n+1)} + \ldots\right\} \quad \text{...........(16)}.$$

324. An important case of our general formulæ occurs when ψ represents a disturbance which is propagated wholly *outwards*. At a great distance from the origin, $f_n(i\kappa r) = f_n(-i\kappa r) = 1$, and thus, if we restore the time factor ($e^{i\kappa at}$), we have

$$r\psi_n = S_n e^{i\kappa(at-r)} + S_n' e^{i\kappa(at+r)} \quad \text{..................(1)},$$

of which the second part represents a disturbance travelling inwards. Under the circumstances contemplated we are therefore to take $S_n' = 0$, and thus

$$r\psi_n = S_n f_n(i\kappa r) e^{i\kappa(at-r)} \quad \text{.........................(2)},$$

which represents in the most general manner the n^{th} harmonic component of a disturbance of the given period diffusing itself outwards into infinite space.

The origin of the disturbance may be in a prescribed normal motion of the surface of a sphere of radius c. Let us suppose that at any point on the sphere the outward velocity is represented by $Ue^{i\kappa at}$, U being in general a function of the position of the point considered.

If U be expanded in the spherical harmonic series

$$U = U_0 + U_1 + U_2 + \ldots + U_n + \ldots\ldots\ldots\ldots\ldots(3),$$

we must have by (13) § 323

$$U_n = -\frac{S_n}{c^2} e^{-i\kappa c} F_n (i\kappa c)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The complete value of ψ is thus

$$\psi = -\frac{c^2}{r} e^{i\kappa(at-r+c)} \Sigma \frac{U_n}{F_n (i\kappa c)} f_n (i\kappa r)\ldots\ldots\ldots\ldots(5),$$

where the summation is to be extended to all (integral) values of n. The real part of this equation will give the velocity potential due to the normal velocity $U\cos \kappa at$[1] at the surface of the sphere $r = c$.

Prof. Stokes has applied this solution to the explanation of a remarkable experiment by Leslie, according to which it appeared that the sound of a bell vibrating in a partially exhausted receiver is diminished by the introduction of hydrogen. This paradoxical phenomenon has its origin in the augmented wave-length due to the addition of hydrogen, in consequence of which the bell loses its hold (so to speak) on the surrounding gas. The general explanation cannot be better given than in the words of Prof. Stokes :

"Suppose a person to move his hand to and fro through a small space. The motion which is occasioned in the air is almost exactly the same as it would have been if the air had been an incompressible fluid. There is a mere local reciprocating motion, in which the air immediately in front is pushed forward, and that immediately behind impelled after the moving body, while in the anterior space generally the air recedes from the encroachment of the moving body, and in the posterior space generally flows in from all sides to supply the vacuum which tends to be created; so that in lateral directions the flow of the fluid is backwards, a

[1] The assumption of a real value for U is equivalent to limiting the normal velocity to be in the same phase all over the sphere $r=c$. To include the most general aerial motion U would have to be treated as complex.

portion of the excess of fluid in front going to supply the deficiency behind. Now conceive the periodic time of the motion to be continually diminished. Gradually the alternation of movement becomes too rapid to permit of the full establishment of the merely local reciprocating flow; the air is sensibly compressed and rarefied, and a sensible sound wave (or wave of the same nature, in case the periodic time be beyond the limits suitable to hearing) is propagated to a distance. The same takes place in any gas; and the more rapid be the propagation of condensations and rarefactions in the gas, the more nearly will it approach, in relation to the motions we have under consideration, to the condition of an incompressible fluid; the more nearly will the conditions of the displacement of the gas at the surface of the solid be satisfied by a merely local reciprocating flow."

In discussing the solution (5), Prof. Stokes goes on to say,

"At a great distance from the sphere the function $f_n(i\kappa r)$[1] becomes ultimately equal to 1, and we have

$$\psi = -\frac{c^2}{r} e^{i\kappa(at-r+c)} \Sigma \frac{U_n}{F_n(i\kappa c)} \qquad \text{(6)}.$$

"It appears $\left(\text{from the value of } \frac{d\psi}{dr}\right)$ that the component of the velocity along the radius vector is of the order r^{-1}, and that in any direction perpendicular to the radius vector of the order r^{-2}, so that the lateral motion may be disregarded except in the neighbourhood of the sphere.

"In order to examine the influence of the lateral motion in the neighbourhood of the sphere, let us compare the actual disturbance at a great distance with what it would have been if all lateral motion had been prevented, suppose by infinitely thin conical partitions dividing the fluid into elementary canals, each bounded by a conical surface having its vertex at the centre.

"On this supposition the motion in any canal would evidently be the same as it would be in all directions if the sphere vibrated by contraction and expansion of the surface, the same all round, and such that the normal velocity of the surface was the same as it is at the particular point at which the canal in question abuts on the surface. Now if U were constant the expansion of U would

[1] I have made some slight changes in Prof. Stokes' notation.

be reduced to its first term U_0, and seeing that $f_0(i\kappa r) = 1$, we should have from (5),

$$\psi = -\frac{c^2}{r} \epsilon^{i\kappa(at-r+c)} \frac{U_0}{F_0(i\kappa c)}.$$

This expression will apply to any particular canal if we take U_0 to denote the normal velocity at the sphere's surface for that particular canal; and therefore to obtain an expression applicable at once to all the canals, we have merely to write U for U_0. To facilitate a comparison with (5) and (6), I shall, however, write ΣU_n for U. We have then,

$$\psi = -\frac{c^2}{r} \epsilon^{i\kappa(at-r+c)} \frac{\Sigma U_n}{F_0(i\kappa c)} \qquad\qquad (7).$$

It must be remembered that this is merely an expression applicable at once to all the canals, the motion in each of which takes place wholly along the radius vector, and accordingly the expression is not to be differentiated with respect to θ or ω with the view of finding the transverse velocities.

"On comparing (7) with the expression for the function ψ in the actual motion at a great distance from the sphere (6), we see that the two are identical with the exception that U_n is divided by two different constants, namely $F_0(i\kappa c)$ in the former case and $F_n(i\kappa c)$ in the latter. The same will be true of the leading terms (or those of the order r^{-1}) in the expressions for the condensation and velocity. Hence if the mode of vibration of the sphere be such that the normal velocity of its surface is expressed by a Laplace's function of any one order, the disturbance at a great distance from the sphere will vary from one direction to another according to the same law as if lateral motions had been prevented, the amplitude of excursion at a given distance from the centre varying in both cases as the amplitude of excursion, in a normal direction, of the surface of the sphere itself. The only difference is that expressed by the symbolic ratio $F_n(i\kappa c) : F_0(i\kappa c)$. If we suppose $F_n(i\kappa c)$ reduced to the form $\mu_n(\cos\alpha_n + i\sin\alpha_n)$, the amplitude of vibration in the actual case will be to that in the supposed case as μ_0 to μ_n, and the phases in the two cases will differ by $\alpha_0 - \alpha_n$.

"If the normal velocity of the surface of the sphere be not expressible by a single Laplace's Function, but only by a series, finite or infinite, of such functions, the disturbance at a given

great distance from the centre will no longer vary from one direction to another according to the same law as the normal velocity of the surface of the sphere, since the modulus μ_n and likewise the amplitude α_n of the imaginary quantity $F_n(i\kappa c)$ vary with the order of the function.

"Let us now suppose the disturbance expressed by a Laplace's function of some one order, and seek the numerical value of the alteration of intensity at a distance, produced by the lateral motion which actually exists.

"The intensity will be measured by the *vis viva* produced in a given time, and consequently will vary as the density multiplied by the velocity of propagation multiplied by the square of the amplitude of vibration. It is the last factor alone that is different from what it would have been if there had been no lateral motion. The amplitude is altered in the proportion of μ_0 to μ_n, so that if $\mu_n^2 : \mu_0^2 = I_n$, I_n is the quantity by which the intensity that would have existed if the fluid had been hindered from lateral motion has to be divided.

"If λ be the length of the sound-wave corresponding to the period of the vibration, $\kappa = 2\pi\lambda^{-1}$, so that κc is the ratio of the circumference of the sphere to the length of a wave. If we suppose the gas to be air and λ to be 2 feet, which would correspond to about 550 vibrations in a second, and the circumference $2\pi c$ to be 1 foot (a size and pitch which would correspond with the case of a common house-bell), we shall have $\kappa c = \frac{1}{2}$. The following table gives the values of the squares of the modulus and of the

κc	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	
4	17	16·25	14·879	13·848	20·177	Values of μ_n^2.
2	5	5	9·3125	80	1495·8	
1	2	5	89	8965	300137	
0·5	1·25	16·25	1330·2	236191	72086371	
0·25	1·0625	64·062	20878	14837899	18160×10^6	
4	1	0·95588	0·87523	0·81459	1·1869	Values of I_n.
2	1	1	1·8625	16	299·16	
1	1	2·5	44·5	1982·5	150068	
0·5	1	13	1064·2	188958	57669097	
0·25	1	60·294	19650	18965×10^3	17092×10^6	

ratio I_n for the functions $F_n(i\kappa c)$ of the first five orders, for each of the values 4, 2, 1, $\frac{1}{2}$, and $\frac{1}{4}$ of κc. It will presently appear why

the table has been extended further in the direction of values greater than $\frac{1}{2}$ than it has in the opposite direction. Five significant figures at least are retained.

"When $\kappa c = \infty$ we get from the analytical expressions $I_n = 1$. We see from the table that when κc is somewhat large I_n is liable to be *a little* less than 1, and consequently the sound to be *a little* more intense than if lateral motion had been prevented. The possibility of that is explained by considering that the waves of condensation spreading from those compartments of the sphere which at a given moment are vibrating positively, *i.e.* outwards, after the lapse of a half period may have spread over the neighbouring compartments, which are now in their turn vibrating positively, so that these latter compartments in their outward motion work against a somewhat greater pressure than if such compartment had opposite to it only the vibration of the gas which it had itself occasioned; and the same explanation applies *mutatis mutandis* to the waves of rarefaction. However, the increase of sound thus occasioned by the existence of lateral motion is but small in any case, whereas when κc is somewhat small I_n increases enormously, and the sound becomes a mere nothing compared with what it would have been had lateral motion been prevented.

"The higher be the order of the function, the greater will be the number of compartments, alternately positive and negative as to their mode of vibration at a given moment, into which the surface of the sphere will be divided. We see from the table that for a given periodic time as well as radius the value of I_n becomes considerable when n is somewhat high. However practically vibrations of this kind are produced when the elastic sphere executes, not its principal, but one of its subordinate vibrations, the pitch corresponding to which rises with the order of vibration, so that κ increases with that order. It was for this reason that the table was extended from $\kappa c = 0{\cdot}5$ further in the direction of high pitch than low pitch, namely, to three octaves higher and only one octave lower.

"When the sphere vibrates symmetrically about the centre, i. e. so that any two opposite points of the surface are at a given moment moving with equal velocities in opposite directions, or more generally when the mode of vibration is such that there is no change of position of the centre of gravity of the volume, there

is no term of order 1. For a sphere vibrating in the manner of a bell the principal vibration is that expressed by a term of the order 2, to which I shall now more particularly attend.

"Putting, for shortness, $\kappa^2 c^2 = q$, we have

$$\mu_0^2 = q + 1, \quad \mu_2^2 = (q^{\frac{1}{2}} + 9q^{-\frac{1}{2}})^2 + (4 - 9q^{-1})^2 = q - 2 + 9q^{-1} + 81q^{-2},$$

$$I_2 = \frac{q^3 - 2q^2 + 9q + 81}{q^2(q+1)}.$$

"The minimum value of I_2 is determined by

$$q^3 - 6q^2 - 84q - 54 = 0,$$

giving approximately,

$$q = 12{\cdot}859, \qquad \kappa c = 3{\cdot}586, \qquad \mu_0^2 = 13{\cdot}859, \qquad \mu_2^2 = 12{\cdot}049,$$

$$I_2 = {\cdot}86941;$$

so that the utmost increase of sound produced by lateral motion amounts to about 15 per cent.

"I now come more particularly to Leslie's experiments. Nothing is stated as to the form, size, or pitch of his bell; and even if these had been accurately described, there would have been a good deal of guess-work in fixing on the size of the sphere which should be considered the best representative of the bell. Hence all we can do is to choose such values for κ and c as are comparable with the probable conditions of the experiment.

"I possess a bell, belonging to an old bell-in-air apparatus, which may probably be somewhat similar to that used by Leslie. It is nearly hemispherical, the diameter is 1·96 inch, and the pitch an octave above the middle c of a piano. Taking the number of vibrations 1056 per second, and the velocity of sound in air 1100 feet per second, we have $\lambda = 12{\cdot}5$ inches. To represent the bell by a sphere of the same radius would be very greatly to underrate the influence of local circulation, since near the mouth the gas has but a little way to get round from the outside to the inside or the reverse. To represent it by a sphere of half the radius would still apparently be to underrate the effect. Nevertheless for the sake of rather under-estimating than exaggerating the influence of the cause here investigated, I will make these two suppositions successively, giving respectively $c = {\cdot}98$ and $c = {\cdot}49$, $\kappa c = {\cdot}4926$, and $\kappa c = {\cdot}2463$ for air.

"If it were not for lateral motion the intensity would vary from gas to gas in the proportion of the density into the velocity of propagation, and therefore as the pressure into the square root of the density under a standard pressure, if we take the factor depending on the development of heat as sensibly the same for the gases and gaseous mixtures with which we have to deal. In the following Table the first column gives the gas, the second the

Gas.	p	D	Q_r	$c = \cdot 93$			$c = \cdot 49$		
				q	I_2	Q	q	I_2	Q
Air	1	1	1	·2427	1136	1	·06067	20890	1
Hydrogen	1	·0690	·2627	·01674	284700	·001048	·004186	4604000	·001191
Air rarefied	·01	·01	·01	·2427	1136	·01	·06067	20890	·01
The same filled with *H.*	1	·0783	·2798	·01900	220600	·001440	·004751	3572000	·001637
Air of same density	·0783	·0783	·0783	·2427	1136	·0783	·06067	20890	·0783
Air rarefied ½	·5	·5	·5	·2427	1136	·5	·06067	20890	·5
The same filled with *H.*	1	·5345	·7311	·1297	4322	·1921	·0324	74890	·2039

pressure p, in atmospheres, the third the density D under the pressure p, referred to the density of the air at the atmospheric pressure as unity, the fourth, Q_r, what would have been the intensity had the motion been wholly radial, referred to the intensity in air at atmospheric pressure as unity, or, in other words, a quantity varying as $p \times$ (the density at pressure 1)$^{\frac{1}{2}}$. Then follow the values of q, I_2, and Q, the last being the actual intensity referred to air as before.

"An inspection of the numbers contained in the columns headed Q will shew that the cause here investigated is amply sufficient to account for the facts mentioned by Leslie."

The importance of the subject, and the masterly manner in which it has been treated by Prof. Stokes, will probably be thought sufficient to justify this long quotation. The simplicity of the true explanation contrasts remarkably with conjectures that had previously been advanced. Sir J. Herschel, for example, thought that the mixture of two gases tending to propagate sound with different velocities might produce a confusion resulting in a rapid stifling of the sound.

325. The term of zero order

$$\psi_0 = \frac{S_0}{r} e^{i\kappa(at-r)} \qquad\qquad (1),$$

where S_0 is a complex constant, corresponds to the potential of a *simple source* of arbitrary intensity and phase, situated at the centre of the sphere (§ 279). If, as often happens in practice, the source of sound be a solid body vibrating without much change of volume, this term is relatively deficient. In the case of a rigid sphere vibrating about a position of equilibrium, the deficiency is absolute[1], inasmuch as the whole motion will then be represented by a term of order 1; and whenever the body is very small in comparison with the wave-length, the term of zero order must be insignificant. For if we integrate the equation of motion, $\nabla^2\psi + \kappa^2\psi = 0$, over the small volume included between the body and a sphere closely surrounding it, we see that the whole quantity of fluid which enters and leaves this space is small, and that therefore there is but little total flow across the surface of the sphere.

[1] The centre of the sphere being the origin of coordinates.

Putting $n = 1$, we get for the term of the first order

$$r\psi_1 = S_1 e^{i\kappa(at-r)} \left\{1 + \frac{1}{i\kappa r}\right\} \qquad \text{.......................(2)},$$

and S_1 is proportional to the cosine of the angle between the direction considered and some fixed axis. This expression is of the same form as the potential of a *double* source (§ 294), situated at the centre, and composed of two equal and opposite simple sources lying on the axis in question, whose distance apart is infinitely small, and intensities such that the product of the intensities and distance is finite. For, if x be the axis, and the cosine of the angle between x and r be μ, it is evident that the potential of the double source is proportional to

$$\frac{d}{dx}\left(\frac{e^{-i\kappa r}}{r}\right) = \mu \frac{d}{dr}\left(\frac{e^{-i\kappa r}}{r}\right) = -i\kappa \frac{\mu e^{-i\kappa r}}{r}\left\{1 + \frac{1}{i\kappa r}\right\}.$$

It appears then that the disturbance due to the vibration of a sphere as a rigid body is the same as that corresponding to a double source at the centre whose axis coincides with the line of the sphere's vibration.

The reaction of the air on a small sphere vibrating as a rigid body with a harmonic motion, may be readily calculated from preceding formulæ. If $\dot{\xi}$ denote the velocity of the sphere at time t,

$$U_1 e^{i\kappa at} = \dot{\xi}\mu \qquad \text{..........................(3)},$$

and therefore for the value of ψ at the surface of the sphere, we have from (5) § 324,

$$\psi = -i\kappa a c \dot{\xi}\mu \frac{f_1(i\kappa c)}{F_1(i\kappa c)} \qquad \text{.....................(4)}.$$

The force Ξ due to aerial pressures accelerating the motion is given by

$$\Xi = -\iint \mu\,\delta p\, dS = \rho \iint \mu \dot{\psi}\, dS$$

$$= -i\kappa a\rho c^3 \ddot{\xi} \frac{f_1(i\kappa c)}{F_1(i\kappa c)} \int 2\pi\mu^2 d\mu = -i\kappa a \frac{4\pi c^3}{3} \rho \ddot{\xi} \frac{f_1(i\kappa c)}{F_1(i\kappa c)}.$$

If we write

$$\frac{f_1(i\kappa c)}{F_1(i\kappa c)} = p - iq \qquad \text{........................(5)},$$

then

$$\Xi = -p\frac{4\pi c^3\rho}{3}\ddot{\xi} - q\kappa a\frac{4\pi c^3\rho}{3}\dot{\xi} \qquad \text{...............(6)},$$

inasmuch as

$$\ddot{\xi} = i\kappa a \dot{\xi}.$$

The operation of the air is therefore to increase the effective inertia of the sphere by p times the inertia of the air displaced, and to retard the motion by a force proportional to the velocity, and equal to $\frac{4}{3}\pi\rho c^3 . q\kappa a\dot{\xi}$, these effects being in general functions of the frequency of vibration. By introduction of the values of f_1 and F_1 we find

$$\frac{f_1(i\kappa c)}{F_1(i\kappa c)} = \frac{2+\kappa^2c^2 - i\,\kappa^3c^3}{4+\kappa^4c^4} \quad\text{.................. (7)};$$

so that,

$$p = \frac{2+\kappa^2c^2}{4+\kappa^4c^4}, \qquad q = \frac{\kappa^3c^3}{4+\kappa^4c^4} \quad\text{.................. (8)}.$$

When κc is small, we have approximately $p=\frac{1}{2}$, $q=\frac{1}{4}\kappa^3c^3$. Hence the effective inertia of a small sphere is increased by one-half of that of the air displaced—a quantity independent of the frequency and the same as if the fluid were incompressible. The dissipative term, which corresponds to the energy emitted, is of high order in κc, and therefore (the effects of viscosity being disregarded) the vibrations of a small sphere are but slowly damped.

The motion of an ellipsoid through an incompressible fluid has been investigated by Green[1], and his result is applicable to the calculation of the increase of effective inertia due to a compressible fluid, provided the dimensions of the body be small in comparison with the wave-length of the vibration. For a small circular disc vibrating at right angles to its plane, the increase of effective inertia is to the mass of a sphere of fluid, whose radius is equal to that of the disc, as 2 to π. The result for the case of a sphere given above was obtained by Poisson[2], a short time before the publication of Green's paper.

It has been proved by Maxwell[3] that the various terms of the harmonic expansion of the common potential may be regarded as due to *multiple points* of corresponding degrees of complexity. Thus V_i is proportional to $\dfrac{d^i}{dh_1\,dh_2\ldots dh_i}\left(\dfrac{1}{r}\right)$, where there are i differentiations of r^{-1} with respect to the axes h_1, h_2, &c., any number of which may in particular cases coincide. It might perhaps

[1] *Edinburgh Transactions*, Dec. 16, 1833. Also Green's *Mathematical Papers*, edited by Ferrers. Macmillan & Co., 1871.

[2] *Mémoires de l'Académie des Sciences*, Tom. XI. p. 521.

[3] Maxwell's *Electricity and Magnetism*, Ch. IX.

have been expected that a similar law would hold for the velocity potential with the substitution of $r^{-1}e^{-i\kappa r}$ for r^{-1}. This however is not the case; it may be shewn that the potential of a quadruple source, denoted by $\frac{d^2}{dh_1 dh_2} \cdot \frac{e^{-i\kappa r}}{r}$, corresponds in general not to the term of the second order simply, viz., $S_2 \frac{e^{-i\kappa r}}{r} f_2(i\kappa r)$, but to a combination of this with a term of zero order. The analogy therefore holds only in the single instance of the *double* point or source, though of course the function $r^{-1}e^{-i\kappa r}$ after any number of differentiations continues to satisfy the fundamental equation

$$(\nabla^2 + \kappa^2)\psi = 0.$$

It is perhaps worth notice that the disturbance outside any imaginary sphere which completely encloses the origin of sound may be represented as due to the normal motion of the surface of any smaller concentric sphere, or, as a particular case when the radius of the sphere is infinitely small, as due to a source concentrated in one point at the centre. This source will in general be composed of a combination of multiple sources of all orders of complexity.

326. When the origin of the disturbance is the vibration of a rigid body parallel to its axis of revolution, the various spherical harmonics S_n reduce to simple multiples of the zonal harmonic $P_n(\mu)$, which may be defined as the coefficient of e^n in the expansion of $\{1 - 2e\mu + e^2\}^{-\frac{1}{2}}$ in rising powers of e. And whenever the solid, besides being symmetrical about an axis, is also symmetrical with respect to an equatorial plane (whose intersection with the axis is taken as origin of co-ordinates), the expansion of the resulting disturbance in spherical harmonics will contain terms of odd order only. For example, if the vibrating body were a circular disc moving perpendicularly to its plane, the expansion of ψ would contain terms proportional to $P_1(\mu)$, $P_3(\mu)$, $P_5(\mu)$, &c. In the case of the sphere, as we have seen, the series reduces absolutely to its first term, and this term will generally be preponderant.

On the other hand we may have a vibrating system symmetrical about an axis and with respect to an equatorial plane, but in such a manner that the motions of the parts on the two sides of the plane are opposed. Under this head comes the ideal tuning

fork, composed of equal spheres or parallel circular discs, whose distance apart varies periodically. Symmetry shews that the velocity-potential, being the same at any point and at its image in the plane of symmetry, must be an even function of μ, and therefore expressible by a series containing only the even functions $P_0(\mu)$, $P_2(\mu)$, &c. The second function $P_2(\mu)$ would usually preponderate, though in particular cases, as for example if the body were composed of two discs very close together in comparison with their diameter, the symmetrical term of zero order might become important. A comparison with the known solution for the sphere whose surface vibrates according to any law, will in most cases furnish material for an estimate as to the relative importance of the various terms.

327. The total emission of energy by a vibrating sphere is found by multiplying the variable part of the pressure (proportional to ψ) by the normal velocity and integrating over the surface (§ 245). In virtue of the conjugate property the various spherical harmonic terms may be taken separately without loss of generality. We have (§ 323)

$$\left.\begin{aligned}\psi_n &= i\kappa a\frac{S_n\, e^{i\kappa(at-r)}}{r} f_n(i\kappa r)\\ \frac{d\psi_n}{dr} &= -\frac{S_n\, e^{i\kappa(at-r)}}{r^2} F_n(i\kappa r)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ (1),$$

or on rejecting the imaginary part

$$\left.\begin{aligned}\psi_n &= -\frac{\kappa a S_n}{r}\{\beta'\cos\kappa(at-r)+\alpha'\sin\kappa(at-r)\}\\ \frac{d\psi_n}{dr} &= -\frac{S_n}{r^2}\{\alpha\cos\kappa(at-r)-\beta\sin\kappa(at-r)\}\end{aligned}\right\}\ldots\ldots\ (2),$$

where
$$F=\alpha+i\beta,\qquad f=\alpha'+i\beta'\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Thus
$$\iint\psi_n\frac{d\psi_n}{dr}dS=\iint\psi_n\frac{d\psi_n}{dr}\,.\,r^2d\sigma$$

$$=\frac{\kappa a}{r}\iint S_n^2\,d\sigma\,\{\alpha\beta'\cos^2\kappa(at-r)-\alpha'\beta\sin^2\kappa(at-r)$$
$$+(\alpha\alpha'-\beta\beta')\sin\kappa(at-r)\cos\kappa(at-r)\}.$$

When this is integrated over a long range of time, the periodic terms may be omitted, and thus

$$\int.\iint\psi_n\frac{d\psi_n}{dr}dS\,.\,dt=\frac{\kappa at}{2r}(\alpha\beta'-\alpha'\beta)\iint S_n^2\,d\sigma\ \ldots\ldots\ (4).$$

Now, since there can be on the whole no accumulation of energy in the space included between two concentric spherical surfaces, the rates of transmission of energy across these surfaces must be the same, that is to say $r^{-1}(\alpha'\beta - \beta'\alpha)$ must be independent of r. In order to determine the constant value, we may take the particular case of r indefinitely great, when

$$F_n(i\kappa r) = i\kappa r, \qquad \alpha = 0, \qquad \beta = \kappa r,$$

$$f_n(i\kappa r) = 1, \qquad \alpha' = 1, \qquad \beta' = 0.$$

Thus $$\alpha'\beta - \beta'\alpha = \kappa r, \text{ identically} \quad\quad (5).$$

It may be observed that the left-hand member of (5) when multiplied by i is the imaginary part of $(\alpha + i\beta)(\alpha' - i\beta')$ or of $F_n(i\kappa r) f_n(-i\kappa r)$, so that our result may be expressed by saying that the imaginary part of $F_n(i\kappa r) f_n(-i\kappa r)$ is $i\kappa r$, or

$$F_n(i\kappa r) f_n(-i\kappa r) - F_n(-i\kappa r) f_n(i\kappa r) = 2i\kappa r \quad\quad (6).$$

In this form we shall have occasion presently to make use of it.

The same conclusion may be arrived at somewhat more directly by an application of Helmholtz's theorem (§ 294), i.e. that if two functions u and v satisfy through a closed space S the equation $(\nabla^2 + \kappa^2) u = 0$, then

$$\iint\left(u\frac{dv}{dn} - v\frac{du}{dn}\right) dS = 0 \quad\quad (7).$$

If we take for S the space between two concentric spheres, making

$$u = \frac{S_n e^{-i\kappa r} f_n(i\kappa r)}{r}, \qquad v = \frac{S_n e^{+i\kappa r} f_n(-i\kappa r)}{r},$$

we find that $r^{-1}\{F_n(i\kappa r) f_n(-i\kappa r) - F_n(-i\kappa r) f_n(i\kappa r)\}$ must be independent of r.

We have therefore

$$\int . \iint \psi_n \frac{d\psi_n}{dr} dS . dt = -\tfrac{1}{2}\kappa^2 at \iint S_n^2 d\sigma;$$

so that the expression for the energy emitted in time t is (since $\delta p = -\rho\dot\psi$)

$$W = \tfrac{1}{2}\kappa^2 \rho at \iint S_n^2 d\sigma \quad\quad (8).$$

It will be more instructive to exhibit W as a function of the normal motion at the surface of a sphere of radius c. From (2)

$$\frac{d\psi_n}{dr} = -\frac{S_n}{c^2}\left[\cos \kappa at\,(\alpha \cos \kappa c + \beta \sin \kappa c) + \sin \kappa at\,(\alpha \sin \kappa c - \beta \cos \kappa c)\right],$$

so that, if the amplitude of $\frac{d\psi_n}{dr}$ be U_n, we have as the relation between S_n and U_n

$$c^4 U_n^2 = (\alpha^2 + \beta^2) S_n^2 \quad \text{.........................(9)}.$$

Thus
$$W = \frac{\kappa^2 c^4 \rho a t}{2(\alpha^2+\beta^2)} \iint U_n^2\, d\sigma \quad \text{....................(10)}.$$

This formula may be verified for the particular cases $n=0$ and $n=1$, treated in §§ 280, 325 respectively.

328. If the source of disturbance be a normal motion of a small part of the surface of the sphere ($r=c$) in the immediate neighbourhood of the point $\mu=1$, we must take in the general solution applicable to divergent waves, viz.

$$\psi = -\frac{c^2}{r} e^{i\kappa(at-r+c)} \Sigma \frac{U_n}{F_n(i\kappa c)} f_n(i\kappa r) \quad \text{.................(1)},$$

$$U_n = \tfrac{1}{2}(2n+1)\, P_n(\mu) \,.\int_{-1}^{+1} U P_n(\mu)\, d\mu$$

$$= \tfrac{1}{2}(2n+1) P_n(\mu) \int_{-1}^{+1} U d\mu = \frac{2n+1}{4\pi c^2} \,.\, P_n(\mu) \iint U dS \quad \text{......(2)};$$

for where U is sensible, $P_n(\mu)=1$. Thus

$$\psi = -\frac{e^{i\kappa(at-r+c)}}{4\pi r} \,.\iint U dS \,.\, \Sigma(2n+1) P_n(\mu) \frac{f_n(i\kappa r)}{F_n(i\kappa c)} \quad \text{......(3)}.$$

In this formula $\iint U dS$ measures the intensity of the source.

If $i\kappa c$ be very small,

$$\frac{f_0(i\kappa r)}{F_0(i\kappa c)} = 1 - i\kappa c + \dots, \qquad \frac{f_1(i\kappa r)}{F_1(i\kappa c)} = i\kappa c\left(1 + \frac{1}{i\kappa r}\right) + \dots \text{ \&c.};$$

so that ultimately

$$\psi = -\frac{e^{i\kappa(at-r)}}{4\pi r} \iint U dS \quad \text{.........................(4)},$$

and the waves diverge as from a simple source of equal magnitude.

We will now examine the problem when κc is not very small, taking for simplicity the case where ψ is required at a great distance only, so that $f_n(i\kappa r) = 1$. The factor on which the relative intensities in various directions depend is

$$\Sigma \frac{(2n+1)}{2} \frac{P_n(\mu)}{F_n(i\kappa c)} \quad \text{.........................(5)},$$

and a complete solution of the question would involve a discussion of this series as a function of μ and κc.

Thus, if

$$\Sigma \frac{(2n+1)}{2} \frac{P_n(\mu)}{F_n(i\kappa c)} = F + iG \quad \text{....................(6)},$$

$$\psi = -\frac{1}{2\pi r} \iint U dS \,.\, \{F^2 + G^2\}^{\frac{1}{2}} \,.\, e^{i\kappa(at - r + c) + i\theta} \quad \text{............(7)},$$

where

$$\tan\theta = G : F \quad \text{...........................(8)}.$$

The intensity of the vibrations in the various directions is thus measured by $F^2 + G^2$. If, as before, $F_n = \alpha + i\beta$,

$$\left.\begin{aligned} F &= \Sigma \frac{2n+1}{2} \frac{\alpha P_n(\mu)}{\alpha^2+\beta^2} \\ -G &= \Sigma \frac{2n+1}{2} \frac{\beta P_n(\mu)}{\alpha^2+\beta^2} \end{aligned}\right\} \quad \text{.........................(9)}.$$

The following table gives the means of calculating F and G for any value of μ, when $\kappa c = \frac{1}{2}$, 1, or 2. In the last case it is necessary to go as far as $n = 7$ to get a tolerably accurate result, and for larger values of κc the calculation would soon become very laborious. In all problems of this sort the harmonic analysis seems to lose its power when the waves are very small in comparison with the dimensions of bodies.

$\kappa c = \frac{1}{2}$.

n	2α	2β	$(n+\frac{1}{2})\alpha \div (\alpha^2+\beta^2)$	$(n+\frac{1}{2})\beta \div (\alpha^2+\beta^2)$
0	+ 2	+ 1	+ ·4	+ ·2
1	+ 4	− 7	+ ·1846153	− ·3230768
2	− 64	− 35	− ·0601891	− ·0328885
3	− 466	+ 853	− ·0034527	+ ·0063201
4	+ 14902	+ 8141	+ ·0004653	+ ·0002542
5	+ 175592	− 321419	+ ·0000144	− ·0000264

$\kappa c = 1.$

n	α	β	$(n+\frac{1}{2})\alpha \div (\alpha^2+\beta^2)$	$(n+\frac{1}{2})\beta \div (\alpha^2+\beta^2)$
0	+ 1	+ 1	+ ·25	+ ·25
1	+ 2	− ·1	+ ·6	− ·3
2	− 5	− 8	− ·140449	− ·224719
3	− 53	+ 34	− ·046784	+ ·030013
4	+ 296	+ 461	+ ·004438	+ ·006912
5	+ 4951	− 3179	+ ·000787	− ·000505
6	− 40613	− 63251	− ·000047	− ·000073
7	− 936340	+ 601217	− ·000006	+ ·000004

$\kappa c = 2.$

n	α	β	$(n+\frac{1}{2})\alpha \div (\alpha^2+\beta^2)$	$(n+\frac{1}{2})\beta \div (\alpha^2+\beta^2)$
0	+ 1	+ 2	+ ·1	+ ·2
1	+ 2	+ 1	+ ·6	+ ·3
2	+ 1·75	− 2·5	+ ·46980	− ·67114
3	− 8	− 4	− ·35	− ·175
4	− 16·1875	+ 35·125	− ·04870	+ ·10567
5	+ 186·625	+ 85·4375	+ ·02436	+ ·01115
6	+ 538·80	− 1177·3	+ ·00209	− ·00456
7	− 8621·7	− 3945·8	− ·00072	− ·00033

The most interesting question on which this analysis informs us is the influence which a rigid sphere, situated close to the source, has on the intensity of sound in different directions. By the principle of reciprocity (§ 294) the source and the place of observation may be interchanged. When therefore we know the relative intensities at two distant points B, B', due to a source A on the surface of the sphere, we have also the relative intensities (measured by potential) at the point A, due to distant sources at B and B'. On this account the problem has a double interest.

As a numerical example I have calculated the values of $F + iG$ and $F^2 + G^2$ for the above values of κc, when $\mu = 1$, $\mu = -1$, $\mu = 0$, that is, looking from the centre of the sphere, in the direction of the source, in the opposite direction, and laterally.

When κc is zero, the value of $F^2 + G^2$ is ·25, which therefore represents on the same scale as in the table the intensity due to an unobstructed source of equal magnitude. We may interpret κc

as the ratio of the circumference of the sphere to the wave-length of the sound.

κc	μ	$F+iG$	F^2+G^2
$\frac{1}{2}$	1	·521503 + ·149417i	·294291
	− 1	·159149 − ·484149i	·259729
	0	·130244 − ·216539i	·231999
1	1	·667938 + ·238369i	·502961
	− 1	− ·140055 − ·302609i	·285220
	0	+ ·321903 − ·364974i	·236828
2	1	·79683 + ·23421i	·6898
	− 1	·24954 + ·50586i	·3182
	0	− ·15381 − ·57662i	·3562

In looking at these figures the first point which attracts attention is the comparatively slight deviation from uniformity in the intensities in different directions. Even when the circumference of the sphere amounts to twice the wave-length, there is scarcely anything to be called a sound shadow. But what is perhaps still more unexpected is that in the first two cases the intensity behind the sphere exceeds that in a transverse direction. This result depends mainly on the preponderance of the term of the first order, which vanishes with μ. The order of the more important terms increases with κc; when κc is 2, the principal term is that of the second order.

Up to a certain point the augmentation of the sphere will increase the total energy emitted, because a simple source emits twice as much energy when close to a rigid plane as when entirely in the open. Within the limits of the table this effect masks the obstruction due to an increasing sphere, so that when $\mu = -1$, the intensity is greater when the circumference is twice the wave-length than when it is half the wave-length, the source itself remaining constant.

If the source be not simple harmonic with respect to time, the relative proportions of the various constituents will vary to some extent both with the size of the sphere, and with the direction of the point of observation, illustrating the fundamental character of the analysis into simple harmonics.

When κc is decidedly less than one-half, the calculation may be conducted with sufficient approximation algebraically. The result is

$$F^2 + G^2 = \tfrac{1}{4} + \tfrac{1}{12}\kappa^2 c^2 \left(\tfrac{7}{4}\mu^2 - \tfrac{4}{3}\right)$$

$$+ \tfrac{1}{4}\kappa^4 c^4 \left(1 + \tfrac{3}{2}\mu + \tfrac{50}{81}P_2 + \tfrac{25}{81}P_2^2 - \tfrac{7}{20}\mu P_3 + \tfrac{6}{175}P_4\right)$$

$$+ \text{terms in } \kappa^6 c^6 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

It appears that so far as the term in $\kappa^2 c^2$, the intensity is an even function of μ, viz. the same at any two points diametrically opposed. For the principal directions $\mu = \pm 1$, or 0, the numerical calculation of the coefficient of $\kappa^4 c^4$ is easy on account of the simple values then assumed by the functions P. Thus

$$(\mu = 1), \quad F^2 + G^2 = \tfrac{1}{4} + \tfrac{5}{144}\kappa^2 c^2 + \cdot 77755\,\kappa^4 c^4 + \ldots\ldots$$

$$(\mu = -1), \quad F^2 + G^2 = \tfrac{1}{4} + \tfrac{5}{144}\kappa^2 c^2 + \cdot 02755\,\kappa^4 c^4 + \ldots\ldots$$

$$(\mu = 0), \quad F^2 + G^2 = \tfrac{1}{4} - \tfrac{1}{9}\kappa^2 c^2 + \cdot 19534\,\kappa^4 c^4 + \ldots\ldots$$

When $\kappa^4 c^4$ can be neglected, the intensity is less in a lateral direction than immediately in front of or behind the sphere. Or, by the reciprocal property, a source at a distance will give a greater intensity on the surface of a small sphere at the point furthest from the source than in a lateral position.

If we apply these formulæ to the case of $\kappa c = \tfrac{1}{2}$, we get

$$(\mu = 1), \quad F^2 + G^2 = \cdot 3073,$$

$$(\mu = -1), \quad F^2 + G^2 = 2604,$$

$$(\mu = 0), \quad F^2 + G^2 = \cdot 2344,$$

which agree pretty closely with the results of the more complete calculation.

For other values of μ, the coefficient of $\kappa^4 c^4$ in (10) might be calculated with the aid of tables of Legendre's functions, or from the following algebraic expression in terms of μ[1],

$$1 + \tfrac{3}{2}\mu + \tfrac{50}{81}P_2 + \tfrac{25}{81}P_2^2 - \tfrac{7}{20}\mu P_3 + \tfrac{6}{175}P_4$$

$$= \cdot 78138 + 1\cdot 5\,\mu + \cdot 85938\,\mu^2 - \cdot 03056\,\mu^4.$$

The *difference* of intensities in the directions $\mu = +1$ and $\mu = -1$ may be very simply expressed. Thus

[1] For the forms of the functions P, see § 334.

$$(F^2+G^2)_{\mu=1}-(F^2+G^2)_{\mu=-1}=\tfrac{3}{4}\,\kappa^4c^4.$$

$$\text{If } \kappa c=\tfrac{3}{8},\quad \tfrac{3}{4}\,\kappa^4c^4=\cdot 0148.$$

$$\text{If } \kappa c=\tfrac{2}{8},\quad \tfrac{3}{4}\,\kappa^4c^4=\cdot 0029.$$

$$\text{If } \kappa c=\tfrac{1}{8},\quad \tfrac{3}{4}\,\kappa^4c^4=\cdot 0002.$$

At the same time the total value of F^2+G^2 approximates to $\cdot 25$, when κc is small.

These numbers have an interesting bearing on the explanation of the part played by the two ears in the perception of the quarter from which a sound proceeds.

It should be observed that the variations of intensity in different directions about which we have been speaking are due to the presence of the sphere as an obstacle, and not to the fact that the source is on the circumference of the sphere instead of at the centre. At a great distance a small displacement of a source of sound will affect the *phase* but not the *intensity* in any direction.

In order to find the alteration of phase we have for a small sphere

$$F=\tfrac{1}{2}+\kappa^2c^2\left(-\tfrac{1}{2}+\tfrac{3}{4}\mu-\tfrac{5}{18}P_2\right),\quad G=\kappa c\left(-\tfrac{1}{2}+\tfrac{3}{4}\mu\right),$$

$$\tan\theta=G:F=\kappa c\left(-1+\tfrac{3}{2}\mu\right),\ \text{ or }\ \theta=\kappa c\left(-1+\tfrac{3}{2}\mu\right)\ \text{ nearly.}$$

Thus in (7)

$$e^{i\kappa(at-r+c)+i\theta}=e^{i\kappa(at-r+\frac{3}{2}\mu c)},$$

from which we may infer that the phase at a distance is the same as if the source had been situated at the point $\mu=1$, $r=\tfrac{3}{2}c$ (instead of $r=c$), and there had been no obstacle.

329. The functional symbols f and F may be expressed in terms of P. It is known[1] that

$$P_n(\mu)=1-\frac{n}{1}\cdot\frac{n+1}{1}\cdot\frac{1-\mu}{2}+\ldots\ldots$$

or, changing μ into $1-\mu$,

$$P_n(1-\mu)=1-\frac{n}{1}\cdot\frac{n+1}{1}\cdot\frac{\mu}{2}+\frac{n(n-1)}{1\,.\,2}\cdot\frac{(n+1)(n+2)}{1\,.\,2}\cdot\frac{\mu^2}{2^2}-\ldots(1).$$

Consider now the symbolic operator $P_n\left(1-\dfrac{d}{dy}\right)$, and let it operate on y^{-1}.

[1] Thomson and Tait's *Nat. Phil.* § 782 (quoted from Murphy).

Since $$\left(\frac{d}{dy}\right)^s \cdot \frac{1}{y} = (-1)(-2) \ldots\ldots (-s)\, y^{-s-1};$$

$$P_n\left(1-\frac{d}{dy}\right)\cdot\frac{1}{y} = y^{-1} + \frac{n(n+1)}{1.2}\, y^{-2} + \frac{(n-1)\ldots\ldots(n+2)}{2.4}\, y^{-3} + \ldots\ldots$$

A comparison with (9) § 323 now shews that

$$f_n(y) = y P_n\left(1-\frac{d}{dy}\right)\cdot\frac{1}{y} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

from which we deduce by a known formula,

$$\frac{e^{-y}}{y} f_n(y) = e^{-y} P_n\left(1-\frac{d}{dy}\right)\frac{1}{y} = (-1)^n P_n\left(\frac{d}{dy}\right)\cdot\frac{e^{-y}}{y} \quad\ldots\ldots(3).$$

In like manner,

$$\frac{e^{+y}}{y} f_n(-y) = P_n\left(\frac{d}{dy}\right)\cdot\frac{e^{+y}}{y}.$$

If we now identify y with $i\kappa r$, we see that the general solution, (12) § 323, may be written

$$\psi_n = (-1)^n i\kappa S_n P_n\left(\frac{d}{d.i\kappa r}\right)\cdot\frac{e^{-i\kappa r}}{i\kappa r} + i\kappa S_n' P_n\left(\frac{d}{d.i\kappa r}\right)\cdot\frac{e^{+i\kappa r}}{i\kappa r} \ldots(4),$$

from which the second term is to be omitted, if no part of the disturbance be propagated inwards.

Again from (14) § 323 we see that

$$\frac{F_n(y)}{y^2} = \left(1-\frac{d}{dy}\right)\cdot\frac{f_n(y)}{y},$$

whence $$F_n(y) = y^2 P_n\left(1-\frac{d}{dy}\right)\left(1-\frac{d}{dy}\right)\cdot\frac{1}{y} \ldots\ldots\ldots\ldots(5),$$

and $$\frac{F_n(y)\, e^{-y}}{y^2} = -(-)^n P_n\left(\frac{d}{dy}\right)\frac{d}{dy}\cdot\frac{e^{-y}}{y} \ldots\ldots\ldots\ldots\ldots (6).$$

Similarly, $$\frac{F_n(-y)\, e^{+y}}{y^2} = -P_n\left(\frac{d}{dy}\right)\frac{d}{dy}\cdot\frac{e^{+y}}{y} \ldots\ldots\ldots\ldots\ldots(7).$$

Using these expressions in (13) § 323, we get

$$\frac{d\psi_n}{dr} = (-)^{n+1}\kappa^2 S_n P_n\left(\frac{d}{d.i\kappa r}\right)\frac{d}{d.i\kappa r}\cdot\frac{e^{-i\kappa r}}{i\kappa r}$$
$$-\kappa^2 S_n' P_n\left(\frac{d}{d.i\kappa r}\right)\frac{d}{d.i\kappa r}\cdot\frac{e^{+i\kappa r}}{i\kappa r} \ldots\ldots\ldots (8).$$

330. We have already considered in some detail the form assumed by our general expressions when there is no source at infinity. An equally important class of cases is defined by the condition that there be no source at the origin. We shall now investigate what restriction is thereby imposed on our general expressions.

Reversing the series for f_n, we have

$$r\psi_n = \frac{1.3.5\ldots(2n-1)}{(i\kappa r)^n}\{S_n e^{-i\kappa r}(1 + i\kappa r + \ldots) \\ + (-1)^n S_n' e^{+i\kappa r}(1 - i\kappa r + \ldots)\},$$

shewing that, as r diminishes without limit, $r\psi_n$ approximates to

$$r\psi_n = \frac{1.3.5\ldots(2n-1)}{(i\kappa r)^n}\{S_n + (-1)^n S_n'\}.$$

In order therefore that ψ_n may be finite at the origin,

$$S_n + (-1)^n S_n' = 0 \quad\ldots\ldots\ldots\ldots(1)$$

is a necessary condition; that it is sufficient we shall see later.

Accordingly (12) § 323 becomes

$$r\psi_n = S_n\{e^{-i\kappa r} f_n(i\kappa r) - (-1)^n e^{+i\kappa r} f_n(-i\kappa r)\} \quad\ldots\ldots(2).$$

If, separating the real and imaginary parts of f_n, we write (as before)

$$f_n = \alpha' + i\beta' \quad\ldots\ldots\ldots\ldots(3),$$

(2) may be put into the form

$$r\psi_n = -2i^{n+1} S_n\{\alpha' \sin(\kappa r + \tfrac{1}{2}n\pi) - \beta' \cos(\kappa r + \tfrac{1}{2}n\pi)\} \quad\ldots\ldots(4).$$

Another form may be derived from (4) § 329. We have

$$\psi_n = -2i\kappa(-1)^n S_n P_n\left(\frac{d}{d.i\kappa r}\right).\frac{e^{+i\kappa r} - e^{-i\kappa r}}{2i\kappa r}$$

$$= -2i\kappa(-1)^n S_n P_n\left(\frac{d}{d.i\kappa r}\right).\frac{\sin\kappa r}{\kappa r} \quad\ldots\ldots\ldots(5).$$

Since the function P_n is either wholly odd or wholly even, the expression for ψ_n is wholly real or wholly imaginary.

In order to prove that the value of ψ_n in (5) remains finite when r vanishes, we begin by observing that

$$\frac{2\sin\kappa r}{\kappa r}=\int_{-1}^{+1}e^{-i\kappa r\mu}\,d\mu \quad\text{..................... (6)},$$

so that
$$2P_n\left(\frac{d}{d\,.\,i\kappa r}\right)\frac{\sin\kappa r}{\kappa r}=\int_{-1}^{+1}P_n\left(\frac{d}{d\,.\,i\kappa r}\right)e^{i\kappa r\mu}\,d\mu$$

$$=\int_{-1}^{+1}P_n(\mu)\,e^{i\kappa r\mu}\,d\mu \quad\text{............... (7)},$$

as is obvious when it is considered that the effect of differentiating $e^{i\kappa r\mu}$ any number of times with respect to $i\kappa r$ is to multiply it by the corresponding power of μ. It remains to expand the expression on the right in ascending powers of r. We have

$$\int_{-1}^{+1}P_n(\mu)\,e^{i\kappa r\mu}\,d\mu=\int_{-1}^{+1}d\mu\,P_n(\mu)\left\{1+i\kappa r\,.\,\mu+\frac{(i\kappa r)^2}{1\,.\,2}\,.\,\mu^2+\dots\right.$$
$$\left.+\frac{(i\kappa r)^n}{1\,.\,2\dots n}\,.\,\mu^n+\dots\right\}.$$

Now any positive integral power of μ, such as μ^p, can be expanded in a terminating series of the functions P, the function of highest order being P_p. It follows that, if $p<n$,

$$\int_{-1}^{+1}\mu^p\,P_n(\mu)\,d\mu=0,$$

by known properties of these functions; so that the lowest power of $i\kappa r$ in $\int_{-1}^{+1}P_n(\mu)\,e^{i\kappa r\mu}\,d\mu$ is $(i\kappa r)^n$. Retaining only the leading term, we may write

$$\int_{-1}^{+1}P_n(\mu)\,e^{i\kappa r\mu}\,d\mu=\frac{(i\kappa r)^n}{1\,.\,2\dots n}\int_{-1}^{+1}\mu^n P_n(\mu)\,d\mu.$$

From the expression for $P_n(\mu)$ in terms of μ, viz.

$$P_n(\mu)=\frac{1\,.\,3\,.\,5\dots(2n-1)}{1\,.\,2\,.\,3\dots n}\left\{\mu^n-\frac{n(n-1)}{2(2n-1)}\mu^{n-2}\right.$$
$$\left.+\frac{n(n-1)(n-2)(n-3)}{2\,.\,4\,.\,(2n-1)(2n-3)}\mu^{n-4}-\dots\right\}\quad\text{............ (8)},$$

we see that

$$\mu^n=\frac{1\,.\,2\,.\,3\dots n}{1\,.\,3\,.\,5\dots(2n-1)}P_n(\mu)+\text{terms in }\mu\text{ of lower order than }\mu^n;$$

and therefore

$$\int_{-1}^{+1} \mu^n P_n(\mu)\,d\mu = \frac{1.2.3\ldots n}{1.3.5\ldots(2n-1)}\int_{-1}^{+1}[P_n(\mu)]^2\,d\mu$$

$$= \frac{1.2.3\ldots n}{1.3.5\ldots(2n-1)}\cdot\frac{2}{2n+1}\ \ldots\ldots\ldots\ldots\ldots(9).$$

Accordingly, by (5) and (7)

$$\psi_n = -2i\kappa(-1)^n S_n \frac{(i\kappa r)^n}{1.3.5\ldots(2n+1)} + \ldots\ \ldots\ldots\ldots(10),$$

which shews that ψ_n vanishes with r, except when $n=0$.

The complete series for ψ_n, when there is no source at the pole, is more conveniently obtained by the aid of the theory of Bessel's functions. The differential equations (4) § 200, satisfied by these functions, viz.

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{m^2}{z^2}\right)y = 0\ \ldots\ldots\ldots\ldots\ldots\ldots(11),$$

may also be written in the form

$$\frac{d^2(yz^{\frac{1}{2}})}{dz^2} + \left(1 - \frac{4m^2-1}{4z^2}\right)yz^{\frac{1}{2}} = 0\ \ldots\ldots\ldots\ldots(12).$$

It is known (§ 200) that the solution of (11) subject to the condition of finiteness when $z=0$, is $y = A\,J_m(z)$, where

$$J_m(z) = \frac{z^m}{2^m\,\Gamma(m+1)}\left\{1 - \frac{z^2}{2.(2m+2)} + \frac{z^4}{2.4.(2m+2)(2m+4)} - \ldots\right\}\ldots\ldots\ldots\ldots(13),$$

is the Bessel's function of order m.

When m is integral, $\Gamma(m+1) = 1.2.3\ldots m$; but here we have to do with m fractional and of the form $n+\frac{1}{2}$, n being integral. In this case

$$\Gamma(m+1) = \frac{1.3.5\ldots(2n+1)}{2^{n+1}}\cdot\sqrt{\pi}\ \ldots\ldots\ldots\ldots\ldots(14).$$

Referring now to (12), we see that the solution of

$$\frac{d^2\theta}{dz^2} + \left(1 - \frac{4m^2-1}{4z^2}\right)\theta = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

under the same condition of finiteness when $z=0$, is

$$\theta = Az^{\frac{1}{2}}J_m(z)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16).$$

Now the function ψ_n, with which we are at present concerned, satisfies (4) § 323, viz.

$$\frac{d^2(r\psi_n)}{d(\kappa r)^2}+\left(1-\frac{n(n+1)}{(\kappa r)^2}\right)r\psi_n=0 \quad \text{............(17)},$$

which is of the same form as (15), if $m=n+\frac{1}{2}$; so that the solution is

$$\psi_n=A(\kappa r)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa r)$$

$$=A\frac{(\kappa r)^n\sqrt{2}}{1.3\ldots(2n+1)\sqrt{\pi}}\left\{1-\frac{(\kappa r)^2}{2.(2n+3)}+\frac{(\kappa r)^4}{2.4.(2n+3)(2n+5)}-\ldots\right\} \quad \text{..........(18)}.$$

Determining the constant by a comparison with (10), we find

$$\psi_n=-2(-1)^n i^{n+1}\kappa S_n\left(\frac{\pi}{2\kappa r}\right)^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa r)$$

$$=-2i\kappa(-1)^n S_n\frac{(i\kappa r)^n}{1.3.5\ldots(2n+1)}\left\{1-\frac{\kappa^2r^2}{2(2n+3)}+\frac{\kappa^4r^4}{2.4.(2n+3)(2n+5)}-\frac{\kappa^6r^6}{2.4.6.(2n+3)(2n+5)(2n+7)}+\ldots\right\}$$

$$\text{..............(19)},$$

as the complete expression for ψ_n in rising powers of r.

Comparing the different expressions (5) and (19) for ψ_n, we obtain

$$P_n\left(\frac{d}{d.i\kappa r}\right).\frac{\sin\kappa r}{\kappa r}=i^n\left(\frac{\pi}{2\kappa r}\right)^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa r) \quad \text{............(20)}.$$

If $F=\alpha+i\beta$, the corresponding expressions for $\frac{d\psi_n}{dr}$, are

$$\frac{d\psi_n}{dr}=-\frac{S_n}{r^2}\{e^{-i\kappa r}F_n'(i\kappa r)-(-1)^n e^{+i\kappa r}F_n'(-i\kappa r)\}$$

$$=\frac{2i^{n+1}S_n}{r^2}\{\alpha\sin(\kappa r+\tfrac{1}{2}n\pi)-\beta\cos(\kappa r+\tfrac{1}{2}n\pi)\}$$

$$=-2i\kappa^2(-1)^n S_n P_n\left(\frac{d}{d.i\kappa r}\right)\frac{d}{d.\kappa r}.\frac{\sin\kappa r}{\kappa r}$$

$$=\frac{2n(-1)^n\kappa^2 S_n(i\kappa r)^{n-1}}{1.3.5\ldots(2n+1)}\left\{1-\frac{n+2}{2n(2n+3)}\kappa^2r^2+\ldots\right\} \quad \text{......(21)}.$$

It will be convenient to write down for reference the forms of ψ and $\frac{d\psi}{dr}$ for the first three orders.

$$n=0 \quad \begin{cases} \psi_0 = -2i\kappa\, S_0 \dfrac{\sin \kappa r}{\kappa r}, \\[2mm] \dfrac{d\psi_0}{dr} = \dfrac{2i\kappa S_0}{r}\left\{\dfrac{\sin \kappa r}{\kappa r} - \cos \kappa r\right\}. \end{cases}$$

$$n=1 \quad \begin{cases} \psi_1 = \dfrac{2S_1}{r}\left\{\cos \kappa r - \dfrac{\sin \kappa r}{\kappa r}\right\}, \\[2mm] \dfrac{d\psi_1}{dr} = -\dfrac{2S_1}{r^2}\left\{2\cos \kappa r + \left(\kappa r - \dfrac{2}{\kappa r}\right)\sin \kappa r\right\}. \end{cases}$$

$$n=2 \quad \begin{cases} \psi_2 = -\dfrac{2iS_2}{r}\left\{\left(1 - \dfrac{3}{\kappa^2 r^2}\right)\sin \kappa r + \dfrac{3}{\kappa r}\cos \kappa r\right\}, \\[2mm] \dfrac{d\psi_2}{dr} = \dfrac{2iS_2}{r^2}\left\{\left(4 - \dfrac{9}{\kappa^2 r^2}\right)\sin \kappa r - \left(\kappa r - \dfrac{9}{\kappa r}\right)\cos \kappa r\right\}. \end{cases}$$

331. One of the most interesting applications of these results is to the investigation of the motion of a gas within a rigid spherical envelope. To determine the free periods we have only to suppose that $\frac{d\psi}{dr}$ vanishes, when r is equal to the radius of the envelope. Thus in the case of the symmetrical vibrations, we have to determine κ,

$$\tan \kappa r = \kappa r \quad\text{.............................(1)},$$

an equation which we have already considered in the chapter on membranes, § 207. The first finite root ($\kappa r = 1{\cdot}4303\,\pi$) corresponds to the symmetrical vibration of lowest pitch. In the case of a higher root, the vibration in question has spherical nodes, whose radii correspond to the inferior roots.

Any cone, whose vertex is at the origin, may be made rigid without affecting the conditions of the question.

The loops, or places of no pressure variation, are given by $(\kappa r)^{-1}\sin \kappa r = 0$, or $\kappa r = m\pi$, where m is any integer, except zero.

The case of $n=1$, when the vibrations may be called diametral, is perhaps the most interesting. S_1, being a harmonic of order 1, is proportional to $\cos\theta$ where θ is the angle between r

and some fixed direction of reference. Since $\frac{d\psi_1}{d\theta}$ vanishes only at the poles, there are no conical nodes[1] with vertex at the centre. Any meridianal plane, however, is nodal, and may be supposed rigid. Along any specified radius vector, ψ_1 and $\frac{d\psi_1}{d\theta}$ vanish, and change sign, with $\cos \kappa r - (\kappa r)^{-1} \sin \kappa r$, viz. when $\tan \kappa r = \kappa r$. The loops in the present case therefore coincide with the nodal surfaces of the radial vibrations.

To find the spherical nodes, we have

$$\tan \kappa r = \frac{2\kappa r}{2 - \kappa^2 r^2} \quad \text{.........................} (2).$$

The first root is $\kappa r = 0$. Calculating from Trigonometrical Tables by trial and error, I find for the next root, which corresponds to the vibration of most importance within a sphere, $\kappa r = 119{\cdot}26 \times \frac{\pi}{180}$; so that $r : \lambda = {\cdot}3313$.

The air sways from side to side in much the same manner as in a doubly closed pipe. Without analysis we might anticipate that the pitch would be higher for the sphere than for a closed pipe of equal length, because the sphere may be derived from the cylinder with closed ends, by filling up part of the latter with obstructing material, the effect of which must be to sharpen the *spring*, while the mass to be moved remains but little changed. In fact, for a closed pipe of length $2r$,

$$r : \lambda = {\cdot}25.$$

The sphere is thus higher in pitch than the cylinder by about a Fourth.

The vibration now under consideration is the gravest of which the sphere is capable; it is more than an octave graver than the gravest radial vibration. The next vibration of this type is such that $\kappa r = 340{\cdot}35 \frac{\pi}{180}$, or

$$r : \lambda = {\cdot}9454,$$

and is therefore higher than the first radial.

[1] A node is a surface which might be supposed rigid, viz. one across which there is no motion.

When κr is great, the roots of (2) may be conveniently calculated by means of a series. If $\kappa r = m\pi - y$, then

$$\tan y = \frac{2(m\pi - y)}{(m\pi - y)^2 - 2},$$

from which we find

$$\kappa r = m\pi - \frac{2}{m\pi} - \frac{16}{3m^3\pi^3} + \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

When $n = 2$, the general expression for S_n is

$$S_2 = A_0(\cos^2\theta - \tfrac{1}{3}) + (A_1 \cos\omega + B_1 \sin\omega)\sin\theta\cos\theta$$
$$+ (A_2\cos 2\omega + B_2 \sin 2\omega)\sin^2\theta \ldots (4),$$

from which we may select for special consideration the following notable cases:

(α) the zonal harmonic,

$$S_2 = A_0(\cos^2\theta - \tfrac{1}{3}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Here $\frac{d\psi_2}{d\theta}$ is proportional to $\sin 2\theta$, and therefore vanishes when $\theta = \frac{1}{2}\pi$. This shews that the equatorial plane is a nodal surface, so that the same motion might take place within a closed hemisphere. Also since S_2 does not involve ω, any meridianal plane may be regarded as rigid.

(β) the sectorial harmonic

$$S_2 = A_2 \cos 2\omega \sin^2\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Here again $\frac{d\psi_2}{d\theta}$ varies as $\sin 2\theta$, and the equatorial plane is nodal. But $\frac{d\psi_2}{d\omega}$ varies as $\sin 2\omega$, and therefore does not vanish independently of θ, except when $\sin 2\omega = 0$. It appears accordingly that two, and but two, meridianal planes are nodal, and that these are at right angles to one another.

(γ) the tesseral harmonic,

$$S_2 = A_1 \cos\omega \sin\theta \cos\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

In this case $\frac{d\psi_2}{d\theta}$ vanishes independently of ω with $\cos 2\theta$, that is, when $\theta = \frac{1}{4}\pi$, or $\frac{3}{4}\pi$, which gives a nodal cone of revolution whose vertical angle is a right angle. $\frac{d\psi_2}{d\omega}$ varies as $\sin\omega$, and thus there is one meridianal nodal plane, and but one.

The spherical nodes are given by

$$\tan \kappa r = \frac{\kappa^3 r^3 - 9\kappa r}{4\kappa^2 r^2 - 9} \quad\text{.........................(7),}$$

of which the first finite solution is

$$\kappa r = 3\cdot 3422,$$

giving a tone graver than any of the radial group.

In the case of the general harmonic, the equation giving the tones possible within a sphere of radius r may be written (21) § 330

$$\tan (\kappa r + \tfrac{1}{2} n\pi) = \beta : \alpha \quad\text{.........................(8),}$$

or

$$P_n \left(\frac{d}{d\,.\,i\kappa r}\right) \frac{d}{d.\kappa r} \cdot \frac{\sin \kappa r}{\kappa r} = 0 \quad\text{..................(9),}$$

or again,

$$2\,\kappa r\, J'_{n+\frac{1}{2}}(\kappa r) = J_{n+\frac{1}{2}}(\kappa r) \quad\text{..................(10).}$$

Table *A* shews the values of λ for a sphere of radius unity, corresponding to the more important modes of vibration. In *B* is exhibited the frequency of the various vibrations referred to the gravest of the whole system. The Table is extended far enough to include two octaves.

TABLE A,

Giving the values of λ for a sphere of unit radius.

Order of Harmonic.

Number of internal spherical nodes.		0	1	2	3	4	5	6
	0	1·3983	3·0186	1·8800	1·392	1·113	·9300	·8002
	1	·81334	1·0577	·86195	·7320	·6385		
	2	·57622	·68251	·59208	·5248			
	3	·44670	·50653	·45380				
	4	·36485	·40330					
	5	·30833	·33523					

TABLE B.

Pitch of each tone, referred to gravest.	Order of Harmonic.	Number of internal spherical nodes.	Pitch of each tone, referred to gravest.	Order of Harmonic.	Number of internal spherical nodes
1·0000	1	0	2·8540	1	1
1·6056	2	0	3·2458	5	0
2·1588	0	0	3·5021	2	1
2·169	3	0	3·7114	0	1
2·712	4	0	3·772	6	0

332. If we drop unnecessary constants, the particular solution for the vibrations of gas within a spherical case of radius unity is represented by

$$\psi_n = S_n\,(\kappa r)^{-\frac{1}{2}}\,J_{n+\frac{1}{2}}(\kappa r)\cos(\kappa at - \theta)\ldots\ldots\ldots\ldots(1),$$

where κ is a root of

$$2\kappa\,J'_{n+\frac{1}{2}}(\kappa) = J_{n+\frac{1}{2}}(\kappa)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

In generalising this, we must remember that S_n may be composed of several terms, corresponding to each of which there may exist a vibration of arbitrary amplitude and phase. Further, each term in S_n may be associated with any, or all, of the values of κ, determined by (2). For example, under the head of $n=2$, we might have

$$\psi_2 = A\,(\cos^2\theta - \tfrac{1}{3})\,(\kappa_1 r)^{-\frac{1}{2}}\,J_{n+\frac{1}{2}}(\kappa_1 r)\cos(\kappa_1 at + \theta_1)$$
$$+\,B\cos 2\omega\sin^2\theta\,(\kappa_2 r)^{-\frac{1}{2}}\,J_{n+\frac{1}{2}}(\kappa_2 r)\cos(\kappa_2 at + \theta_2),$$

κ_1 and κ_2 being different roots of

$$2\kappa\,J'_{\frac{5}{2}}(\kappa) = J_{\frac{5}{2}}(\kappa).$$

Any two of the constituents of ψ are conjugate, *i.e.* will vanish, when multiplied together, and integrated over the volume of the sphere. This follows from the property of the spherical harmonics, wherever the two terms considered correspond to different values of n, or to two different constituents of S_n[1]. The only case remaining for consideration requires us to shew that

$$\int_0^1 r^2dr\,.\,(\kappa_1 r)^{-\frac{1}{2}}\,J_{n+\frac{1}{2}}(\kappa_1 r)\,.\,(\kappa_2 r)^{-\frac{1}{2}}\,J_{n+\frac{1}{2}}(\kappa_2 r) = 0\ldots..(3),$$

[1] Thomson and Tait's *Nat. Phil.*, p. 151.

where κ_1 and κ_2 are *different* roots of

$$2\kappa J'_{n+\frac{1}{2}}(\kappa) = J_{n+\frac{1}{2}}(\kappa) \quad\text{..........................(4)},$$

which is an immediate consequence of a fundamental property of these functions (§ 203). There is therefore no difficulty in adapting the general solution to prescribed initial circumstances.

In order to illustrate this subject we will take the case, where initially the gas is in its position of equilibrium but is moving with constant velocity parallel to x. This condition of things would be approximately realised, if the case, having been previously in uniform motion, were suddenly stopped.

Since there is no initial condensation or rarefaction, all the quantities θ_n vanish. If $\dfrac{d\psi}{dx}$ be initially unity, we have $\psi = x = r\mu$, which shews that the solution contains only terms of the first order in spherical harmonics. The solution is therefore of the form

$$\psi = A_1(\kappa_1 r)^{-\frac{1}{2}} J_{\frac{3}{2}}(\kappa_1 r)\,\mu\cos\kappa_1 at$$
$$+ A_2(\kappa_2 r)^{-\frac{1}{2}} J_{\frac{3}{2}}(\kappa_2 r)\,\mu\cos\kappa_2 at + \quad\text{..................(5)},$$

where κ_1, κ_2; &c. are roots of

$$2\kappa J_{\frac{3}{2}}'(\kappa) = J_{\frac{3}{2}}(\kappa) \quad\text{.............................(6)}.$$

To determine the coefficients, we have initially for values of r from 0 to 1,

$$r = A_1(\kappa_1 r)^{-\frac{1}{2}} J_{\frac{3}{2}}(\kappa_1 r) + A_2(\kappa_2 r)^{-\frac{1}{2}} J_{\frac{3}{2}}(\kappa_2 r) + \quad\text{.........(7)}.$$

Multiplying by $r^{\frac{3}{2}} J_{\frac{3}{2}}(\kappa r)$ and integrating with respect to r from 0 to 1, we find

$$\int_0^1 r^{\frac{5}{2}} J_{\frac{3}{2}}(\kappa r)\,dr = A\kappa^{-\frac{1}{2}} \int_0^1 [J_{\frac{3}{2}}(\kappa r)]^2 r\,dr \quad\text{...............(8)},$$

the other terms on the right vanishing in virtue of the conjugate property. Now by (16), § 203,

$$2\int_0^1 [J_{\frac{3}{2}}(\kappa r)]^2 r\,dr = [J_{\frac{3}{2}}'(\kappa)]^2 + \left(1 - \frac{9}{4\kappa^2}\right)[J_{\frac{3}{2}}(\kappa)]^2$$
$$= \left(1 - \frac{2}{\kappa^2}\right)[J_{\frac{3}{2}}(\kappa)]^2 \quad\text{.......................(9)},$$

by (6).

The evaluation of $\int_0^1 r^{\frac{3}{2}} J_{\frac{3}{2}}(\kappa r)\,dr$ may be effected by the aid of a general theorem relating to these functions. By the fundamental differential equation

$$\int_0^r r^{n+1}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ_n(\kappa r)}{dr}\right)+\left(\kappa^2-\frac{n^2}{r^2}\right)J_n(\kappa r)\right]dr=0,$$

whence by integration by parts we obtain,

$$\kappa^2\int_0^r r^{n+1}J_n(\kappa r)\,dr = n\,r^n J_n(\kappa r) - r^{n+1}\frac{dJ_n(\kappa r)}{dr} \;\ldots\ldots\ldots\ldots(10),$$

or, if we make $r=1$,

$$\kappa^2\int_0^1 r^{n+1}J_n(\kappa r)\,dr = n\,J_n(\kappa) - \kappa J_n'(\kappa)\ldots\ldots\ldots\ldots\ldots(11).$$

Thus in the case, with which we are here concerned,

$$\kappa^2\int_0^1 r^{\frac{5}{2}}J_{\frac{3}{2}}(\kappa r)\,dr = \tfrac{3}{2}J_{\frac{3}{2}}(\kappa) - \kappa J_{\frac{3}{2}}'(\kappa) = J_{\frac{3}{2}}(\kappa) \quad \text{by (6)}.$$

Equation (8) therefore takes the form

$$A=\frac{2\kappa^{\frac{1}{2}}}{(\kappa^2-2)J_{\frac{3}{2}}(\kappa)}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

and the final solution is

$$\psi=\Sigma\,\frac{2r^{-\frac{1}{2}}\mu}{\kappa^2-2}\,\frac{J_{\frac{3}{2}}(\kappa r)}{J_{\frac{3}{2}}(\kappa)}\cos\kappa at\;\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

where the summation is to be extended to all the admissible values of κ.

When $t=0$, and $r=1$, we must have $\psi=\mu$, and accordingly

$$\Sigma\,\frac{2}{\kappa^2-2}=1\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

It will be remembered that the higher values of κ are approximately, (3) § 331

$$\kappa=m\pi-\frac{2}{m\pi}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15).$$

The first value of κ is 2·0815, and the second 5·9402, whence

$$\frac{2}{\kappa_1^2-2}=\cdot 85742, \qquad \frac{2}{\kappa_2^2-2}=\cdot 06009,$$

shewing that the first term in the series for ψ is by far the most important.

It may be well to recall here that

$$J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\left(\frac{\sin z}{z} - \cos z\right) \quad\text{.....................}(16).$$

Equation (14) may be verified thus; The quantities κ are the roots of

$$\frac{d}{dz}\left\{z^{-\frac{1}{2}} J_{\frac{3}{2}}(z)\right\} = 0,$$

or, if $\phi = z^{-\frac{1}{2}} J_{\frac{3}{2}}(z)$, the roots of $\phi' = 0$, where ϕ satisfies

$$\phi'' + \frac{2}{z}\phi' + \left(1 - \frac{2}{z^2}\right)\phi = 0 \quad\text{.....................}(17).$$

Now, since the leading term in the expansion of ϕ' in ascending powers of z, is independent of z, we may take

$$\phi' = \text{const.}\left\{1 - \frac{z^2}{\kappa_1^2}\right\}\left\{1 - \frac{z^2}{\kappa_2^2}\right\} \ldots\ldots$$

whence, by taking the logarithms and differentiating,

$$-\frac{\phi''}{\phi'} = \frac{2z}{\kappa_1^2 - z^2} + \frac{2z}{\kappa_2^2 - z^2} + \ldots$$

If we now put $z^2 = 2$, we get by (17),

$$\Sigma \frac{2}{\kappa^2 - 2} = -\frac{\phi''}{z\phi'}\ (z^2 = 2) = 1.$$

333. In a similar manner we may treat the problem of the vibrations of air included between rigid concentric spherical surfaces, whose radii are r_1 and r_2. For by (13) § 323, if $\frac{d\psi_n}{dr}$ vanish for these values of r,

$$e^{2i\kappa r_1}\frac{F_n(-i\kappa r_1)}{F_n(+i\kappa r_1)} = e^{2i\kappa r_2}\frac{F_n(-i\kappa r_2)}{F_n(+i\kappa r_2)},$$

whence

$$\tan\kappa(r_1 - r_2) = \frac{\left(\frac{\beta}{\alpha}\right)_1 - \left(\frac{\beta}{\alpha}\right)_2}{1 + \left(\frac{\beta}{\alpha}\right)_1\left(\frac{\beta}{\alpha}\right)_2} \quad\text{........................}(1),$$

where as before

$$F_n(+i\kappa r) = \alpha + i\beta \quad\text{..............................}(2).$$

When the difference between r_1 and r_2 is very small compared with either, the problem identifies itself with that of the vibration of a spherical sheet of air, and is best solved independently. In (1) § 323, if ψ be independent of r, as it is evident that it must approximately be in the case supposed, we have

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi}{d\theta}\right)+\frac{1}{\sin^2\theta}\frac{d^2\psi}{d\omega^2}+\kappa^2r^2\psi=0 \quad\text{..............(3)},$$

whose solution is simply

$$\psi_n=S_n \quad\text{..............................(4)},$$

while the admissible values of κ^2 are given by

$$\kappa^2r^2=n(n+1) \quad\text{........................(5)}.$$

The interval between the gravest tone ($n=1$) and the next is such that two of them would make a twelfth (octave + fifth). The problem of the spherical sheet of gas will be further considered in the following chapter.

334. The next application that we shall make of the spherical harmonic analysis is to investigate the disturbance which ensues when plane waves of sound impinge on an obstructing sphere. Taking the centre of the sphere as origin of polar co-ordinates, and the direction from which the waves come as the axis of μ, let ϕ be the potential of the unobstructed plane waves. Then, leaving out an unnecessary complex coefficient, we have

$$\phi=e^{i\kappa(at+x)}=e^{i\kappa at}\,.\,e^{i\kappa r\mu} \quad\text{......................(1)},$$

and the solution of the problem requires the expansion of $e^{i\kappa r\mu}$ in spherical harmonics. On account of the symmetry the harmonics reduce themselves to Legendre's functions $P_n(\mu)$, so that we may take

$$e^{i\kappa r\mu}=A_0+A_1P_1+\ldots+A_nP_n+ \quad\text{...........(2)},$$

where $A_0\ldots$ are functions of r, but not of μ. From what has been already proved we may anticipate that A_n, considered as a function of r, must vary as

$$P_n\left(\frac{d}{d\,.\,i\kappa r}\right)\frac{\sin\kappa r}{\kappa r}, \quad \text{or as } r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa r),$$

but the same result may easily be obtained directly. Multiplying

(2) by $P_n(\mu)$, and integrating with respect to μ from $\mu=-1$ to $\mu=+1$, we find

$$\int_{-1}^{+1} P_n(\mu)\, e^{i\kappa r\mu}\, d\mu = A_n \int_{-1}^{+1} (P_n)^2\, d\mu = \frac{2A_n}{2n+1} \quad \text{.........(3)};$$

and, as in § 330,

$$\int_{-1}^{+1} P_n(\mu)\, e^{i\kappa r\mu}\, d\mu = 2P_n\left(\frac{d}{d\,.\,i\kappa r}\right).\frac{\sin \kappa r}{\kappa r},$$

so that finally

$$\frac{A_n}{2n+1} = P_n\left(\frac{d}{d\,.\,i\kappa r}\right).\frac{\sin \kappa r}{\kappa r} = i^n \sqrt{\frac{\pi}{2\kappa r}}\,.\,J_{n+\frac{1}{2}}(\kappa r) \quad \text{........(4)}.$$

In the problem in hand the whole motion outside the sphere may be divided into two parts; the first, that represented by ϕ and corresponding to undisturbed plane waves, and the second a disturbance due to the presence of the sphere, and radiating outwards from it. If the potential of the latter part be ψ, we have (2) § 324 on replacing the general harmonic S_n by $a_n P_n(\mu)$,

$$\left.\begin{aligned} r\psi_n &= a_n P_n(\mu)\,.\,e^{-i\kappa r} f_n(i\kappa r) \\ r^2 \frac{d\psi_n}{dr} &= -a_n P_n(\mu)\,.\,e^{-i\kappa r} F_n(i\kappa r) \end{aligned}\right\} \quad \text{............(5)}.$$

The velocity potential of the whole motion is found by addition of ϕ and ψ, the constants a_n being determined by the boundary conditions, whose form depends upon the character of the obstruction presented by the sphere. The simplest case is that of a rigid and fixed sphere, and then the condition to be satisfied when $r=c$ is that

$$\frac{d\phi}{dr} + \frac{d\psi}{dr} = 0 \quad \text{..........................(6)},$$

a relation which must of course hold good for each harmonic element separately. For the element of order n, we get

$$a_n = (2n+1)\frac{\kappa c^2 e^{i\kappa c}}{F_n(i\kappa c)} P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{d}{d\,.\,\kappa c}.\frac{\sin \kappa c}{\kappa c} \quad \text{.........(7)}.$$

Corresponding to the plane waves $\phi = e^{i\kappa(at+x)}$, the disturbance due to the presence of the sphere is expressed by

$$\begin{aligned}\psi &= \frac{\kappa c^2}{r} e^{i\kappa(at-r+c)} \\ &\times \Sigma_{n=0}^{n=\infty} \frac{2n+1}{F_n(i\kappa c)} P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{d}{d\,.\,\kappa c}.\frac{\sin \kappa c}{\kappa c}\,.\,P_n(\mu)\,.\,f_n(i\kappa r) \quad \text{...(8)}.\end{aligned}$$

At a sufficient distance from the source of disturbance we may take $f_n(i\kappa r)=1$. In order to pass to the solution of a real problem, we may separate the real and imaginary parts, and throw away the latter. On this supposition the plane waves are represented by

$$[\phi]=\cos\kappa\,(at+x)\text{......................}(9).$$

Confining ourselves for simplicity's sake to parts of space at a great distance from the sphere, where $f_n(i\kappa r)=1$, we proceed to extract the real part of (8). Since the functions P are wholly even or wholly odd,

$$P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{d}{d\,.\,\kappa c}\cdot\frac{\sin\kappa c}{\kappa c}$$

is wholly real or wholly imaginary, so that this factor presents no difficulty. $\{F_n(i\kappa c)\}^{-1}$, however, is complex, and since $F_n(i\kappa c)=\alpha+i\beta$,

$$\{F_n(i\kappa c)\}^{-1}=\frac{\alpha-i\beta}{\alpha^2+\beta^2}=\frac{e^{i\gamma}}{\sqrt{(\alpha^2+\beta^2)}},$$

if $\tan\gamma=-\beta\div\alpha$.

Thus

$$\psi=\Sigma\,(2n+1)\frac{\kappa c^2}{r}\,e^{i\{\kappa(at-r+c)+\gamma\}}$$

$$\times\{\alpha^2+\beta^2\}^{-\frac{1}{2}}P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{d}{d\,.\,\kappa c}\cdot\frac{\sin\kappa c}{\kappa c}\,.\,P_n(\mu)\text{.......}(10).$$

When therefore n is even,

$$[\psi]=(2n+1)\frac{\kappa c^2}{r}\cos\{\kappa(at-r+c)+\gamma\}$$

$$\times\{\alpha^2+\beta^2\}^{-\frac{1}{2}}P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{d}{d\,.\,\kappa c}\cdot\frac{\sin\kappa c}{\kappa c}\,.\,P_n(\mu)\text{....}(11),$$

while, if n be odd,

$$[\psi]=(2n+1)\frac{\kappa c^2}{r}\,i\sin\{\kappa(at-r+c)+\gamma\}$$

$$\times\{\alpha^2+\beta^2\}^{-\frac{1}{2}}P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{d}{d\,.\,\kappa c}\cdot\frac{\sin\kappa c}{\kappa c}\,.\,P_n(\mu)\text{...}(12).$$

As examples we may write down the terms in $[\psi]$, involving harmonics of orders 0, 1, 2. The following table of the functions $P_n(\mu)$ will be useful.

$$P_0 = 1,$$
$$P_1 = \mu,$$
$$P_2 = \tfrac{3}{2}(\mu^2 - \tfrac{1}{3}),$$
$$P_3 = \tfrac{5}{2}(\mu^3 - \tfrac{3}{5}\mu),$$
$$P_4 = \tfrac{35}{8}(\mu^4 - \tfrac{6}{7}\mu^2 + \tfrac{3}{35}),$$
$$P_5 = \tfrac{63}{8}(\mu^5 - \tfrac{10}{9}\mu^3 + \tfrac{5}{21}\mu).$$

We have,

$$n = 0, \quad \alpha^2 + \beta^2 = 1 + \kappa^2 c^2, \quad \tan\gamma_0 = -\kappa c,$$

$$[\psi_0] \quad \frac{\kappa c^2}{r}\{1 + \kappa^2 c^2\}^{-\frac{1}{2}} \frac{d}{d.\kappa c}\frac{\sin\kappa c}{\kappa c} . \cos\{\kappa(at - r + c) + \gamma_0\} \ldots\ldots (13),$$

$$n = 1, \quad \alpha^2 + \beta^2 = \kappa^2 c^2 + \frac{4}{\kappa^2 c^2}, \quad \tan\gamma_1 = -\frac{\kappa^2 c^2 - 2}{2\kappa c},$$

$$[\psi_1] = \frac{3\kappa c^2}{r}\left\{\kappa^2 c^2 + \frac{4}{\kappa^2 c^2}\right\}^{-\frac{1}{2}} \frac{d^2}{d(\kappa c)^2} . \frac{\sin\kappa c}{\kappa c} . \mu . \sin\{\kappa(at - r + c) + \gamma_1\} \ldots\ldots (14),$$

$$n = 2, \quad \alpha^2 + \beta^2 = \kappa^2 c^2 - 2 + \frac{9}{\kappa^2 c^2} + \frac{81}{\kappa^4 c^4}, \quad \tan\gamma_2 = -\frac{\kappa^2 c^2 - 9}{4\kappa^2 c^2 - 9},$$

$$[\psi_2] = -\frac{45\kappa c^2}{r}\left\{\kappa^2 c^2 - 2 + \frac{9}{\kappa^2 c^2} + \frac{81}{\kappa^4 c^4}\right\}^{-\frac{1}{2}}$$
$$\times \left\{\frac{d^2}{d(\kappa c)^2} + \frac{1}{3}\frac{d}{d(\kappa c)}\right\}\frac{\sin\kappa c}{\kappa c} . (\mu^2 - \tfrac{1}{3})\cos\{\kappa(at - r + c) + \gamma_2\} \ldots (15).$$

The solution of the problem here obtained, though analytically quite general, is hardly of practical use except when κc is a small quantity. In this case we may advantageously expand our results in rising powers of κc.

$$[\psi_0] = -\frac{\kappa^2 c^3}{3r}\left(1 - \tfrac{3}{5}\kappa^2 c^2 + \tfrac{3}{7}\kappa^4 c^4 - \tfrac{19}{54}\kappa^6 c^6 + \ldots\right)$$
$$\times \cos\{\kappa(at - r + c) + \gamma_0\} \ldots\ldots\ldots\ldots (16).$$

$$[\psi_1] = -\frac{\kappa^2 c^3}{2r}(1 - \tfrac{3}{10}\kappa^2 c^2 - \tfrac{3}{28}\kappa^4 c^4 + \tfrac{1}{27}\kappa^6 c^6 + \ldots)$$
$$\times \mu . \sin\{\kappa(at - r + c) + \gamma_1\} \ldots\ldots\ldots\ldots (17),$$

$$[\psi_2] = -\frac{\kappa^4 c^5}{9r}(1 - \tfrac{25}{126}\kappa^2 c^2 + \tfrac{13}{567}\kappa^4 c^4 + \ldots)$$
$$\times (\mu^2 - \tfrac{1}{3})\cos\{\kappa(at - r + c) + \gamma_2\} \ldots\ldots (18).$$

It appears that while $[\psi_0]$ and $[\psi_1]$ are of the same order in the small quantity κc, $[\psi_2]$ is two orders higher. We shall find presently that the higher harmonic components in $[\psi]$ depend upon still more elevated powers of κc. For a first approximation, then, we may confine ourselves to the elements of order 0 and 1.

Although $[\psi_0]$ contains a cosine, and $[\psi_1]$ a sine, they nevertheless differ in phase by a small quantity only. Comparing two of the values of $\frac{d\psi_n}{dr}$ in (21) § 330 we see that

$$\alpha \sin(\kappa r + \tfrac{1}{2} n\pi) - \beta \cos(\kappa r + \tfrac{1}{2} n\pi)$$
$$= -(-1)^n \frac{n(\kappa r)^{n+1}}{1.3.5\ldots(2n+1)} + \text{higher powers of } \kappa r$$

identically. Dividing by $\alpha \cos(\kappa r + \frac{1}{2}n\pi)$, we get ultimately

$$\tan(\kappa r + \tfrac{1}{2}n\pi) - \frac{\beta}{\alpha} = -\frac{(-1)^n}{\alpha\cos(\kappa r + \frac{1}{2}n\pi)} \cdot \frac{n(\kappa r)^{n+1}}{1.3.5\ldots(2n+1)}.$$

When n is *even*, this equation becomes on substitution for α of its leading term,

$$\tan \kappa r - \frac{\beta}{\alpha} = -\frac{n}{(n+1)(2n+1)} \frac{(\kappa r)^{2n+1}}{\{1.3.5\ldots(2n-1)\}^2}\ldots(19).$$

For example, if $n = 2$,

$$\tan \kappa r - \left(\frac{\beta}{\alpha}\right)_2 = -\frac{2(\kappa r)^5}{3^2.5} + \ldots\ldots$$

When n is at all high, the expressions $\tan \kappa r$ and $\beta \div \alpha$ become very nearly identical for moderate values of κr.

When n is *odd*, we get in a nearly similar manner,

$$\cot \kappa r + \frac{\beta}{\alpha} = \frac{n(\kappa r)^{2n-1}}{(n+1)(2n+1)\{1.3.5\ldots(2n-1)\}^2} + \ldots\ldots$$

or

$$\tan \kappa r + \frac{\alpha}{\beta} = -\frac{n(\kappa r)^{2n+1}}{(n+1)(2n+1)\{1.3.5\ldots(2n-1)\}^2} + \ldots\ldots(20).$$

From these results we see that when n is even,

$$\gamma = -\kappa c \quad \text{approximately,}$$

and when n is odd,

$$\gamma = \tfrac{1}{2}\pi - \kappa c \quad \text{approximately.}$$

The velocity-potential of the disturbance due to a small rigid and fixed sphere is therefore approximately,

$$[\psi_0]+[\psi_1]=-\frac{\kappa^2c^3}{3r}(1+\tfrac{3}{2}\mu)\cos\kappa(at-r)$$

$$=-\frac{\pi T}{r\lambda^2}(1+\tfrac{3}{2}\mu)\cos\kappa(at-r)\;.....(21),$$

if T denote the volume of the obstacle, the corresponding direct wave being

$$[\phi]=\cos\kappa(at+x)\;.......................(22).$$

For a given obstacle and a given distance the ratio of the amplitudes of the scattered and the direct waves is in general proportional to the inverse square of the wave-length, and the ratio of intensities is proportional to the inverse fourth power (§ 296).

In order to compare the intensities of the primary and scattered sounds, we may suppose the former to originate in a simple source, provided it be sufficiently distant (R) from T. Thus, if

$$[\phi]=\frac{\cos\kappa(at-R)}{R}\;.....................(23),$$

$$[\psi]=-\frac{\pi T}{rR\lambda^2}(1+\tfrac{3}{2}\mu)\cos\kappa(at-r)\;..........(24);$$

so that at equal distances from their sources the secondary and the primary waves are in the ratio

$$-\frac{\pi T}{R\lambda^2}(1+\tfrac{3}{2}\mu)\;.......................(25).$$

The intensities are therefore in the ratio

$$\frac{\pi^2T^2}{R^2\lambda^4}(1+\tfrac{3}{2}\mu)^2\;.....................(26),$$

which, in the case of $\mu=+1$, gives approximately

$$\frac{61{\cdot}72\,T^2}{R^2\lambda^4}\;..........................(27).$$

It must be well understood that in order that this result may apply, λ must be great compared with the linear dimension of T, and R must be great compared with λ.

To find the leading term in the expression for ψ_n, when κc is small, we have in the first place,

$$(2n+1)\,P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{d}{d\,.\,\kappa c}\,.\,\frac{\sin\kappa c}{\kappa c}$$

$$=\frac{n\,i^n\,(\kappa c)^{n-1}}{1\,.\,3\,.\,5\ldots(2n-1)}\left\{1-\frac{(n+2)\,\kappa^2c^2}{2\,.\,n\,.\,(2n+3)}+\ldots\right\}\ldots\ldots(28).$$

Again,

$$\alpha^2+\beta^2=F'_n\,(i\kappa c)\times F'_n\,(-i\kappa c)$$

$$=\{1\,.\,3\,.\,5\ldots(2n-1)\,(n+1)\,(\kappa c)^{-n}\}^2\left\{1+\frac{(n-1)\,\kappa^2c^2}{(n+1)\,(2n-1)}+\ldots\right\}\ldots\ldots\ldots(29);$$

so that

$$\{\alpha^2+\beta^2\}^{-\frac{1}{2}}=\frac{(\kappa c)^n}{1\,.\,3\ldots(2n-1)\,(n+1)}\left\{1-\frac{(n-1)\,\kappa^2c^2}{2\,.\,(n+1)\,(2n-1)}+\ldots\right\}\ldots\ldots\ldots(30).$$

Hence, from (10),

$$\psi_n=\frac{c\,(\kappa c)^{2n}\,n\,i^n\,P_n(\mu)}{r\,\{1\,.\,3\,.\,5\ldots(2n-1)\}^2\,(n+1)}\,e^{i\{\kappa(at-r+c)+\gamma_n\}}$$

$$\times\left\{1-\kappa^2c^2\left(\frac{n-1}{(2n+2)\,(2n-1)}+\frac{n+2}{2n\,(2n+3)}\right)+\ldots\right\}\ldots.(31).$$

When n is even, $\gamma_n=-\kappa c$ approximately, and then

$$[\psi_n]=\frac{c\,(\kappa c)^{2n}\,n\,i^n\,P_n(\mu)}{r\,\{1\,.\,3\ldots\ldots(2n-1)\}^2\,(n+1)}\cos\kappa\,(at-r)$$

$$\times\left\{1-\kappa^2c^2\left(\frac{n-1}{(2n+2)\,(2n-1)}+\frac{n+2}{2n\,(2n+3)}\right)+\ldots\ldots\right\}\ldots\ldots(32),$$

while if n be odd, we have merely to replace i^n by i^{n+1}, the result being then still real.

By means of (31) we may verify the first two terms in the expressions for $[\psi_1]$, $[\psi_2]$, in (17), (18). To the case of $n=0$, (31) does not apply.

Again, by (31),

$$[\psi_3]=\frac{\kappa^6c^7}{120\,r}\{1-\tfrac{77}{540}\kappa^2c^2\}\{\mu^3-\tfrac{3}{5}\mu\}\sin\{\kappa\,(at-r+c)+\gamma_3\}\ldots(33),$$

$$[\psi_4]=\frac{\kappa^8c^9}{3150\,r}\{\mu^4-\tfrac{6}{7}\mu^2+\tfrac{3}{35}\}\cos\{\kappa\,(at-r+c)+\gamma_4\}\ldots\ldots\ldots\ldots(34).$$

Combining (17), (18), (33), (34), we have the value of $[\psi]$ complete as far as the terms which are of the order κ^6c^6 compared

with the two leading terms given in (21). In compounding the partial expressions, it is as necessary to be exact with respect to the phases of the components as with respect to their amplitudes; but for purposes requiring only one harmonic element at a time, the phase is often of subordinate importance. In such cases we may take

$$(n \text{ even})\ \gamma = -\kappa c, \qquad (n \text{ odd})\ \gamma = \tfrac{1}{2}\pi - \kappa c.$$

From (31) or (32) it appears that the leading term in ψ_n rises two orders in κc with each step in the order of the harmonic; and that ψ_n is itself expressed by a series containing only even, or only odd, powers of κc. But besides being of higher order in κc, the leading term becomes rapidly smaller as n increases, on account of the other factors which it contains. This is evident, because for all values of n and μ, $P_n(\mu) < 1$; the same is true of $n \div n+1$; while i^n only affects the phase.

In particular cases any one of the harmonic elements of $[\psi]$ may vanish. From (11), (12) since $\alpha^2 + \beta^2$ cannot vanish, we have in such a case

$$P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{d}{d\,.\,\kappa c}\frac{\sin \kappa c}{\kappa c} = 0,$$

the same equation as that which gives the periods of the vibrations of order n in a closed sphere of radius c. A little consideration will shew that this result might have been expected. The table of § 331 is applicable to this question and shews, among other things, that when κc is small, no harmonic element in $[\psi]$ can vanish.

In consequence of the aerial pressures the sphere is acted on by a force parallel to the axis of μ, whose tendency is to set the sphere into vibration. The magnitude of this force, if σ be the density of the fluid, is given by

$$2\pi c^2\sigma\int_{-1}^{+1}(\dot\phi + \dot\psi)\,\mu\,d\mu,$$

in which, by the conjugate property of Legendre's functions, only the term of the first order affects the result of the integration. Now, when $r = c$,

$$\phi_1 = 3\,e^{i\kappa at}\frac{d}{d\,.\,i\kappa c}\frac{\sin \kappa c}{\kappa c}\,.\,\mu,$$

$$\psi_1 = 3\kappa c\,e^{i\kappa at}\frac{f_1(i\kappa c)}{F_1(i\kappa c)}\frac{d}{d\,.\,i\kappa c}\,.\,\frac{d}{d\,.\,\kappa c}\,.\,\frac{\sin \kappa c}{\kappa c}\,.\,\mu,$$

where

$$f_1(i\kappa c) = 1 + \frac{1}{i\kappa c}, \quad F_1(i\kappa c) = i\kappa c + 2 + \frac{2}{i\kappa c}.$$

In order that the force may vanish, it would be necessary that

$$\frac{d}{d\,.\,\kappa c}\cdot\frac{\sin\kappa c}{\kappa c} + \kappa c\frac{f_1(i\kappa c)}{F_1(i\kappa c)}\frac{d^2}{(d\,.\,\kappa c)^2}\frac{\sin\kappa c}{\kappa c} = 0,$$

which cannot be satisfied by any real value of κc. We conclude that, if the sphere be free to move, it will always be set into vibration.

If instead of being absolutely plane, the primary waves have their origin in a unit source at a great, though finite, distance R from the centre of the sphere, we have

$$\phi = -\frac{1}{4\pi R}e^{i\kappa(at-R)}\Sigma(2n+1)P_n(\mu)\times P_n\left(\frac{d}{d\,.\,i\kappa r}\right)\frac{\sin\kappa r}{\kappa r}\quad\ldots(35),$$

$$\psi = -\frac{\kappa c^2}{4\pi rR}e^{i\kappa(at-R-r+c)}\Sigma(2n+1)\frac{P_n(\mu)f_n(i\kappa r)}{F_n(i\kappa c)}$$

$$\times P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{d}{d\,.\,\kappa c}\frac{\sin\kappa c}{\kappa c}\quad\ldots\ldots\ldots\ldots(36).$$

On the sphere itself $r = c$, so that the value of the total potential at any point at the surface is

$$\phi + \psi = -\frac{e^{i\kappa(at-R)}}{4\pi R}\Sigma(2n+1)P_n(\mu)$$

$$\times\left[P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{\sin\kappa c}{\kappa c} + \kappa c\frac{f_n(i\kappa c)}{F_n(i\kappa c)}P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{d}{d\,.\,\kappa c}\frac{\sin\kappa c}{\kappa c}\right].$$

This expression may be simplified. We have

$$P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{\sin\kappa c}{\kappa c} = \frac{1}{2i\kappa c}\{-(-1)^n e^{-i\kappa c}f_n(i\kappa c) + e^{+i\kappa c}f_n(-i\kappa c)\},$$

$$\frac{d}{d\,.\,\kappa c}\cdot P_n\left(\frac{d}{d\,.\,i\kappa c}\right)\frac{\sin\kappa c}{\kappa c} = \frac{1}{2i\kappa^2c^2}\{(-1)^n e^{-i\kappa c}F_n(i\kappa c) - e^{+i\kappa c}f_n(-i\kappa c)\},$$

and thus the quantity within square brackets may be written

$$\frac{e^{i\kappa c}}{2i\kappa c}\,\frac{F_n(i\kappa c)f_n(-i\kappa c) - F_n(-i\kappa c)f_n(i\kappa c)}{F_n(i\kappa c)},$$

which by (6) § 327 is identical with $e^{i\kappa c}[F_n(i\kappa c)]^{-1}$. Thus

$$\phi + \psi = -\frac{e^{i\kappa(at-R+c)}}{4\pi R}\Sigma(2n+1)\frac{P_n(\mu)}{F_n(i\kappa c)}\quad\ldots\ldots\ldots(37),$$

which is the same as if the source had been on the sphere, and the point at which the potential is required at a great distance (§328), and is an example of the general Principle of Reciprocity. By assuming the principle, and making use of the result (3) of § 328, we see that if the source of the primary waves be at a finite distance R, the value of the total potential at any point on the sphere is

$$\phi+\psi=-\frac{1}{4\pi R}e^{i\kappa(at-R+c)}\Sigma(2n+1)P_n(\mu)\frac{f_n(i\kappa R)}{F_n(i\kappa c)}\text{.....}(38).$$

If A and B be any two points external to the sphere, a unit source at A will give the same total potential at B, as a unit source at B would give at A. In either case the total potential is made up of two parts, of which the first is the same as if there were no obstacle to the free propagation of the waves, and the second represents the disturbance due to the obstacle. Of these two parts the first is obviously the same, whichever of the two points be regarded as source, and therefore the other parts must also be equal, that is the value of ψ at B when A is a source is equal to the value of ψ at A when B is an equal source. Now when the source A is at a great distance R, the value of ψ at a point B whose angular distance from A is $\cos^{-1}\mu$, and linear distance from the centre is r, is (36)

$$\psi=-\frac{\kappa c^2}{4\pi rR}e^{i\kappa(at-R-r+c)}\Sigma(2n+1)\frac{P_n(\mu)f_n(i\kappa r)}{F_n(i\kappa c)}$$
$$\times P_n\left(\frac{d}{d.i\kappa c}\right)\frac{d}{d.\kappa c}.\frac{\sin\kappa c}{\kappa c},$$

and accordingly this is also the value of ψ at a great distance R, when the source is at B. But since ψ is a disturbance radiating outwards from the sphere, its value at any finite distance R may be inferred from that at an infinite distance by introducing into each harmonic term the factor $f_n(i\kappa R)$. We thus obtain the following symmetrical expression

$$\psi=-\frac{\kappa c^2}{4\pi rR}e^{i\kappa(at-R-r+c)}\Sigma(2n+1)\frac{P_n(\mu)}{F_n(i\kappa c)}$$
$$\times f_n(i\kappa R).f_n(i\kappa r)\,P_n\left(\frac{d}{d.i\kappa c}\right)\frac{d}{d.\kappa c}\frac{\sin\kappa c}{\kappa c}\text{..........}(39),$$

which gives this part of the potential at either point, when the other is a unit source.

It should be observed that the general part of the argument does not depend upon the obstacle being either spherical or rigid.

From the expansion of $e^{i\kappa r\mu}$ in spherical harmonics, we may deduce that of the potential of waves issuing from a unit simple source A finitely distant (r) from the origin of co-ordinates. The potential at a point B at an infinite distance R from the origin, and in a direction making an angle $\cos^{-1}\mu$ with r, will be

$$\phi = \frac{e^{-i\kappa(R-\mu r)}}{4\pi R},$$

the time factor being omitted.

Hence by the expansion of $e^{i\kappa r\mu}$

$$\phi = \frac{e^{-i\kappa R}}{4\pi R}\Sigma(2n+1)\,P_n\left(\frac{d}{d\,.\,i\kappa r}\right)\frac{\sin\kappa r}{\kappa r}\,.\,P_n(\mu);$$

from which we pass to the case of a finite R by the simple introduction of the factor $f_n(i\kappa R)$.

Thus the potential at a finitely distant point B of a unit source at A is

$$\phi = \frac{e^{i\kappa(at-R)}}{4\pi R}\Sigma(2n+1)\,P_n\left(\frac{d}{d\,.\,i\kappa r}\right)\frac{\sin\kappa r}{\kappa r}f_n(i\kappa R)\,.\,P_n(\mu)\ldots(40).$$

335. Having considered at some length the case of a rigid spherical obstacle, we will now sketch briefly the course of the investigation when the obstacle is gaseous. Although in all natural gases the compressibility is nearly the same, we will suppose for the sake of generality that the matter occupying the sphere differs in compressibility, as well as in density, from the medium in which the plane waves advance.

Exterior to the sphere, ϕ is the same exactly, and ψ is of the same form as before. For the motion inside the sphere, if $\kappa' = 2\pi \div \lambda'$ be the internal wave-length, (2) § 330,

$$\psi_n = \frac{a_n' P_n}{r}\{e^{-i\kappa' r}f_n(i\kappa' r) - (-1)^n e^{+i\kappa' r}f_n(-i\kappa' r)\},$$

$$\frac{d\psi_n}{dr} = \frac{2a_n' P_n}{r^2}\,.\,i^{n+1}\{\alpha\sin(\kappa' r + \tfrac{1}{2}n\pi) - \beta\cos(\kappa' r + \tfrac{1}{2}n\pi)\},$$

satisfying the condition of continuity through the centre.

If σ, σ' be the natural densities, m, m' the compressibilities,

$$\frac{\kappa'^2}{\kappa^2}=\frac{\sigma'}{\sigma}\cdot\frac{m}{m'}\quad\text{.............................(1)};$$

and the conditions, to be satisfied by each harmonic element separately, are

$$\frac{d\phi}{dr}+\frac{d\psi}{dr}\text{ (outside)}=\frac{d\psi}{dr}\text{ (inside)}\quad\text{...............(2)},$$

$$\sigma\{\phi+\psi\text{ (outside)}\}=\sigma'\psi\text{ (inside)}\quad\text{.............(3)},$$

expressing respectively the equalities of the normal motions and of the pressures on the two sides of the bounding surface. From these equations the complete solution may be worked out; but we will here confine ourselves to finding the value of the leading terms, when κc, $\kappa' c$ are very small.

In this case, when $r=c$,

$$\left.\begin{aligned}\psi_0\text{ (inside)}&=-2i\kappa' a_0'\\ \frac{d\psi_0}{dr}\text{ (inside)}&=\tfrac{2}{3}i\kappa'^3 c a_0'\end{aligned}\right\}\quad\text{.....................(4)},$$

$$\left.\begin{aligned}\phi_0&=1\\ \frac{d\phi_0}{dr}&=-\tfrac{1}{3}\kappa^2 c\end{aligned}\right\}\quad\text{.........................(5)},$$

$$\left.\begin{aligned}\psi_0\text{ (outside)}&=\frac{a_0}{c}\\ \frac{d\psi_0}{dr}\text{ (outside)}&=-\frac{a_0}{c^2}\end{aligned}\right\}\quad\text{....................(6)}.$$

Using these in (2), (3), and eliminating a_0', retaining only the principal term, we find

$$a_0=-\frac{\kappa^2c^3}{3}\cdot\frac{m'-m}{m'}\quad\text{.........................(7)}.$$

In like manner for the term of first order,

$$\left.\begin{aligned}\psi_1\text{ (inside)}&=-\tfrac{2}{3}a_1'\kappa'^2 c\mu\\ \frac{d\psi_1}{dr}\text{ (inside)}&=-\tfrac{2}{3}a_1'\kappa'^2\mu\end{aligned}\right\}\quad\text{.................(8)},$$

$$\left.\begin{aligned}\phi_1&=i\kappa c\mu\\ \frac{d\phi_1}{dr}&=i\kappa\mu\end{aligned}\right\}\quad\text{..........................(9)},$$

$$\left.\begin{aligned} \psi_1 \text{ (outside)} &= \frac{a_1}{i\kappa c^2}\mu \\ \frac{d\psi_1}{dr} \text{ (outside)} &= -\frac{2a_1}{i\kappa c^3}\mu \end{aligned}\right\} \quad\text{.................(10),}$$

which give

$$a_1 = \frac{\kappa^2 c^3 (\sigma - \sigma')}{\sigma + 2\sigma'} \text{.........................(11).}$$

At a distance from the sphere the disturbance due to it is expressed by

$$\begin{aligned} \psi &= \frac{1}{r} e^{i\kappa(at-r)} \{a_0 + a_1\mu\} \\ &= -\frac{\kappa^2 c^3}{3r} e^{i\kappa(at-r)} \left\{\frac{m'-m}{m'} + 3\frac{\sigma'-\sigma}{\sigma+2\sigma'}\mu\right\} \text{..........(12).} \end{aligned}$$

If we introduce the relations

$$T = \frac{4\pi c^3}{3}, \quad \kappa = 2\pi \div \lambda,$$

and throw away the imaginary part, we obtain

$$\psi = -\frac{\pi T}{\lambda^2 r}\left\{\frac{m'-m}{m'} + 3\frac{\sigma'-\sigma}{\sigma+2\sigma'}\mu\right\}\cos\kappa(at-r)\text{.....(13),}$$

as the expression for the most important part of the disturbance, corresponding to (21) § 334 for a fixed rigid sphere. It appears, as might have been expected, that the term of zero order is due to the variation of compressibility, and that of order one to the variation of density.

From (13) we may fall back on the case of a rigid fixed sphere, by making both σ' and m' infinite. It is not sufficient to make σ' by itself infinite, apparently because, if m' at the same time remained finite, $\kappa'c$ would not be small, as the investigation has assumed.

When $m'-m$, $\sigma'-\sigma$ are small, (13) becomes equivalent to

$$\psi = -\frac{\pi T}{\lambda^2 r}\left\{\frac{m'-m}{m} + \frac{\sigma'-\sigma}{\sigma}\mu\right\}\cos\kappa(at-r),$$

corresponding to $\phi = \cos\kappa at$ at the centre of the sphere. This agrees with the result (13) of § 296, in which the obstacle may be of any form.

In actual gases $m' = m$, and the term of zero order disappears. If the gas occupying the spherical space be incomparably lighter than the other gas, $\sigma' = 0$, and

$$\psi = 3\frac{\pi T}{\lambda^2 r}\mu \cos \kappa (at - r) \qquad (14),$$

so that in the term of order one, the effect is twice that of a rigid body, and has the reverse sign.

The greater part of this chapter is taken from two papers by the author "On the vibrations of a gas contained within a rigid spherical envelope," and an "Investigation of the disturbance produced by a spherical obstacle on the waves of sound[1]," and from the paper by Professor Stokes already referred to.

[1] *Math. Society's Proceedings*, March 14, 1872; Nov. 14, 1872.

CHAPTER XVIII.

SPHERICAL SHEETS OF AIR. MOTION IN TWO DIMENSIONS.

336. In a former chapter (§ 135), we saw that a proof of Fourier's theorem might be obtained by considering the mechanics of a vibrating string. A similar treatment of the problem of a spherical sheet of air will lead us to a proof of Laplace's expansion for a function which is arbitrary at every point of a spherical surface.

As in § 333, if ψ is the velocity-potential, the equation of continuity, referred to the ordinary polar co-ordinates θ, ω, takes the form,

$$c^2 \frac{d^2\psi}{dt^2} = a^2 \left\{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2\psi}{d\omega^2}\right\}.$$

Whatever may be the character of the free motion, it can be analysed into a series of simple harmonic vibrations, the nature of which is determined by the corresponding functions ψ, considered as dependent on space. Thus, if $\psi \propto e^{i\kappa at}$, the equation to determine ψ as a function of θ and ω is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2\psi}{d\omega^2} + \kappa^2c^2\psi = 0 \text{ ....(1).}$$

Again, whatever function ψ may be, it can be expanded by Fourier's theorem[1] in a series of sines and cosines of the multiples of ω. Thus

$$\psi = \psi_0 + \psi_1\cos\omega + \psi_1'\sin\omega + \psi_2\cos 2\omega + \psi_2'\sin 2\omega$$
$$+ \ldots\ldots + \psi_s\cos s\omega + \psi_s'\sin s\omega + \ldots\ldots\ldots\text{(2),}$$

[1] We here introduce the condition that ψ recurs after one revolution round the sphere.

where the coefficients $\psi_0, \psi_1 \dots \psi_1', \psi_2' \dots$ are functions of θ only; and by the conjugate property of the circular functions, each term of the series must satisfy the equation independently. Accordingly,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\psi_s}{d\theta}\right)-\frac{s^2\psi_s}{\sin^2\theta}+\kappa^2c^2\psi_s=0\dots\dots\dots(3)$$

is the equation from which the character of ψ_s or ψ_s' is to be determined. This equation may be written in various ways.

In terms of μ (= cos θ),

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\psi_s}{d\mu}\right\}+h^2\psi_s-\frac{s^2}{1-\mu^2}\psi_s=0\ \dots\dots(4);$$

or, if $\nu=\sin\theta$,

$$\nu^2(1-\nu^2)\frac{d^2\psi_s}{d\nu^2}+\nu(1-2\nu^2)\frac{d\psi_s}{d\nu}+\nu^2h^2\psi_s-s^2\psi_s=0\dots(5),$$

where h^2 is written for κ^2c^2.

When the original function ψ is symmetrical with respect to the pole, that is, depends upon latitude only, s vanishes, and the equations simplify. This case we may conveniently take first. In terms of μ,

$$(1-\mu^2)\frac{d^2\psi_0}{d\mu^2}-2\mu\frac{d\psi_0}{d\mu}+h^2\psi_0=0\ \dots\dots\dots\dots(6).$$

The solution of this equation involves two arbitrary constants, multiplying two definite functions of μ, and may be obtained in the ordinary way by assuming an ascending series and determining the exponents and coefficients by substitution. Thus

$$\begin{aligned}\psi_0=A&\left\{1-\frac{h^2}{1.2}\mu^2+\frac{h^2(h^2-2.3)}{1.2.3.4}\mu^4\right.\\&\left.\qquad-\frac{h^2(h^2-2.3)(h^2-4.5)}{1.2.3.4.5.6}\mu^6+\&c.\right\}\\&+B\left\{\mu-\frac{h^2-1.2}{1.2.3}\mu^3+\frac{(h^2-1.2)(h^2-3.4)}{1.2.3.4.5}\mu^5-\&c.\right\}\ \dots(7),\end{aligned}$$

in which A and B are arbitrary constants.

Let us now further suppose that ψ besides being symmetrical round the pole is also symmetrical with respect to the equator (which is accordingly *nodal*), or in other words that ψ is an

even function of the sine of the latitude (μ). Under these circumstances it is clear that B must vanish, and the value of ψ be expressed simply by the first series, multiplied by the arbitrary constant A. This value of the velocity-potential is the logical consequence of the original differential equation and of the two restrictions as to symmetry. The value of h^2 might appear to be arbitrary, but from what we know of the mechanics of the problem, it is certain beforehand that h^2 is really limited to a series of particular values. The condition, which yet remains to be introduced and by which h is determined, is that the original equation is satisfied at the pole itself, or in other words that the pole is not a source; and this requires us to consider the value of the series when $\mu = 1$. Since the series is an even function of μ, if the pole $\mu = +1$ be not a source, neither will be the pole $\mu = -1$. It is evident at once that if h^2 be of the form $n(n+1)$, where n is an even integer, the series terminates, and therefore remains finite when $\mu = 1$; but what we now want to prove is that, if the series remain finite for $\mu = 1$, h^2 is necessarily of the above-mentioned form. By the ordinary rule it appears at once that, whatever be the value of h^2, the ratio of successive terms tends to the limit μ^2, and therefore the series is convergent for all values of μ less than unity. But for the extreme value $\mu = 1$, a higher method of discrimination is necessary.

It is known[1] that the infinite hypergeometrical series

$$1 + \frac{ab}{cd} + \frac{a(a+1)b(b+1)}{c(c+1)d(d+1)} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)\,d(d+1)(d+2)} + \ldots \quad (8)$$

is convergent, if $c+d-a-b$ be greater than 1, and divergent if $c+d-a-b$ be equal to, or less than 1. In the latter case the value of $c+d-a-b$ affords a criterion of the degree of divergency. Of two divergent series of the above form, for which the values of $c+d-a-b$ are different, that one is *relatively* infinite for which the value of $c+d-a-b$ is the smaller.

Our present series (7) may be reduced to the standard form by taking $h^2 = n(n+1)$, where n is not assumed to be integral. Thus

[1] Boole's *Finite Differences*, p. 79.

$$1-\frac{h^2}{1.2}\mu^2+\frac{h^2(h^2-2.3)}{1.2.3.4}\mu^4-\ldots$$

$$=1-\frac{n(n+1)}{1.2}\mu^2+\frac{n(n+1)(n-2)(n+3)}{1.2.3.4}\mu^4-\ldots$$

$$=1+\frac{(-\frac{1}{2}n)(\frac{1}{2}n+\frac{1}{2})}{1.\frac{1}{2}}\mu^2+\frac{(-\frac{1}{2}n)(-\frac{1}{2}n+1)(\frac{1}{2}n+\frac{1}{2})(\frac{1}{2}n+\frac{1}{2}+1)}{1.2.\frac{1}{2}.\frac{3}{2}}\mu^4$$
$$+\ldots\ldots\ldots\ldots(9),$$

which is of the standard form, if

$$a=-\tfrac{1}{2}n,\qquad b=\tfrac{1}{2}n+\tfrac{1}{2},\qquad c=\tfrac{1}{2},\qquad d=1.$$

Accordingly, since $c+d-a-b=1$, the series is divergent for $\mu=1$, *unless it terminate;* and it terminates only when n is an even integer. We are thus led to the conclusion that when the pole is not a source, and ψ_0 is an even function of μ, h^2 must be of the form $n(n+1)$, where n is an even integer.

In like manner, we may prove that when ψ_0 is an odd function of μ, and the poles are not sources, $A=0$, and h^2 must be of the form $n(n+1)$, n being an *odd* integer.

If n be fractional, both series are divergent for $\mu=\pm 1$, and although a combination of them may be found which remains finite at one or other pole, there can be no combination which remains finite at *both* poles. If therefore it be a condition that no point on the surface of the sphere is a source, we have no alternative but to make n integral, and even then we do not secure finiteness at the poles unless we further suppose $A=0$, when n is odd, and $B=0$, when n is even. We conclude that for a complete spherical layer, the only admissible values of ψ, which are functions of latitude only, and proportional to harmonic functions of the time, are included under

$$\psi=CP_n(\mu),$$

where $P_n(\mu)$ is Legendre's function, and n is any odd or even integer. The possibility of expanding an arbitrary function of latitude in a series of Legendre's functions is a necessary consequence of what has now been proved. Any possible motion of the layer of gas is represented by the series

$$\psi=A_0+P_1(\mu)\left(A_1\cos\frac{\sqrt{1.2}\,at}{c}+B_1\sin\frac{\sqrt{1.2}\,at}{c}\right)+\ldots$$
$$+P_n(\mu)\left(A_n\cos\frac{\sqrt{n(n+1)}\,at}{c}+B_n\sin\frac{\sqrt{n(n+1)}\,at}{c}\right)+\ldots\ldots(10).$$

When $t = 0$,

$$\psi = A_0 + A_1 P_1(\mu) + \ldots + A_n P_n(\mu) + \ldots\ldots\ldots(11),$$

and the value of ψ when $t = 0$ is an *arbitrary* function of latitude.

The method that we have here followed has also the advantage of proving the conjugate property,

$$\int_{-1}^{+1} P_n(\mu)\, P_m(\mu)\, d\mu = 0 \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

where n and m are different integers. For the functions $P(\mu)$ are the *normal* functions (§ 94) for the vibrating system under consideration, and accordingly the expression for the kinetic energy can only involve the *squares* of the generalized velocities. If (12) do not hold good, the *products* also of the velocities must enter.

The value of ψ appropriate to a *plane* layer of vibrating gas can of course be deduced as a particular case of the general solution applicable to a spherical layer. Confining ourselves to the case where there is no source at the pole ($\mu = 1$), we have to investigate the limiting form of $\psi = CP_n(\mu)$, where $n(n+1) = \kappa^2 c^2$, when c^2 and n^2 are infinite. At the same time $\mu - 1$ and ν are infinitesimal, and $c\nu$ passes into the plane polar radius (r), so that $n\nu = \kappa r$. For this purpose the most convenient form of $P_n(\mu)$ is that of Murphy[1]:

$$P_n(\cos\theta) = 1 - \frac{n(n+1)}{1^2}\sin^2\frac{\theta}{2} + \frac{(n-1)n(n+1)(n+2)}{1^2 . 2^2}\sin^4\frac{\theta}{2} - \ldots\ldots\ldots\ldots\ldots(13).$$

The limit is evidently

$$\psi = C\left\{1 - \frac{\kappa^2 r^2}{2^2} + \frac{\kappa^4 r^4}{2^2 . 4^2} - \frac{\kappa^6 r^6}{2^2 . 4^2 . 6^2} + \ldots\right\} = CJ_0(\kappa r)\ldots\ldots(14),$$

shewing that the Bessel's function of zero order is an extreme case of Legendre's functions.

When the spherical layer is not complete, the problem requires a different treatment. Thus, if the gas be bounded by walls stretching along two parallels of latitude, the complete integral involving two arbitrary constants will in general be necessary.

[1] Thomson and Tait's *Nat. Phil.* § 782. [$t = \sin^2\frac{1}{2}\theta$, not $4\sin^2\frac{1}{2}\theta$.] Todhunter's *Laplace's Functions*, § 19.

The ratio of the constants and the admissible values of h^2 are to be determined by the two boundary conditions expressing that at the parallels in question the motion is wholly in longitude. The value of μ being throughout numerically less than unity, the series are always convergent.

If the portion of the surface occupied by gas be that included between two parallels of latitude at equal distances from the equator, the question becomes simpler, since then one or other of the constants A and B in (7) vanishes in the case of each normal function.

337. When the spherical area contemplated includes a pole, we have, as in the case of the complete sphere, to introduce the condition that the pole is not a source. For this purpose the solution in terms of ν, i.e. sin θ, will be more convenient.

If we restrict ourselves for the present to the case of symmetry, we have, putting $s=0$ in (5) § 336,

$$\nu(1-\nu^2)\frac{d^2\psi_0}{d\nu^2}+(1-2\nu^2)\frac{d\psi_0}{d\nu}+h^2\nu\psi_0=0\text{.........}(1).$$

One solution of this equation is readily obtained in the ordinary way by assuming an ascending series and substituting in the differential equation to determine the exponents and coefficients. We get[1]

$$\psi_0=A\left\{1+\frac{0.1-h^2}{2^2}\nu^2+\frac{(0.1-h^2)(2.3-h^2)}{2^2.4^2}\nu^4\right.$$
$$\left.+\frac{(0.1-h^2)(2.3-h^2)(4.5-h^2)}{2^2.4^2.6^2}\nu^6+\ldots\right\}\text{......}(2).$$

This value of ψ_0 is the most general solution of (1), subject to the condition of finiteness when $\nu=0$. The complete solution involving two arbitrary constants provides for a source of arbitrary intensity at the pole, in which case the value of ψ_0 is infinite when $\nu=0$. Any solution which remains finite when $\nu=0$ and involves one arbitrary constant, is therefore the most general possible under the restriction that the pole be not a source. Accordingly it is unnecessary for our purpose to complete the solution. The nature of the second function (involving a logarithm of ν) will be illustrated in the particular case of a plane layer to be considered presently.

[1] Heine's *Kugel-functionen*, § 28.

By writing $n(n+1)$ for h^2 the series within brackets becomes

$$1 - \frac{n(n+1)}{2^2}\nu^2 + \frac{(n-2)\,n\,(n+1)\,(n+3)}{2^2\,.\,4^2}\nu^4 - \ldots\ldots \quad (3),$$

or, when reduced to the standard hypergeometrical form,

$$1 + \frac{(-\tfrac{1}{2}n)(\tfrac{1}{2}n+\tfrac{1}{2})}{1\,.\,1}\nu^2 + \frac{(-\tfrac{1}{2}n)(-\tfrac{1}{2}n+1)(\tfrac{1}{2}n+\tfrac{1}{2})(\tfrac{1}{2}n+\tfrac{1}{2}+1)}{1\,.\,2\,.\,1\,.\,2}\nu^4 + \ldots,$$

corresponding to

$$a = -\tfrac{1}{2}n, \quad b = \tfrac{1}{2}n + \tfrac{1}{2}, \quad c = 1, \quad d = 1.$$

Since $c + d - a - b = \frac{3}{2}$, the series converges for all values of ν from 0 to 1 inclusive. To values of θ $(= \sin^{-1}\nu)$ greater than $\frac{1}{2}\pi$ the solution is inapplicable.

When n is an integer, the series becomes identical with Legendre's function $P_n(\mu)$. If the integer be even, the series terminates, but otherwise remains infinite. Thus, when $n = 1$, the series is identical with the expansion of μ, viz. $\sqrt{(1-\nu^2)}$, in powers of ν.

The expression for ψ in terms of ν may be conveniently applied to the investigation of the free symmetrical vibrations of a spherical layer of air, bounded by a small circle, whose radius is less than the quadrant. The condition to be satisfied is simply $\frac{d\psi}{d\nu} = 0$, an equation by which the possible values of h^2, or $\kappa^2 c^2$, are connected with the given boundary value of ν.

Certain particular cases of this problem may be treated by means of Legendre's functions. Suppose, for example, that $n = 6$, so that $h^2 = \kappa^2 c^2 = 42$. The corresponding solution is $\psi = A\,P_6(\mu)$. The greatest value of μ for which $\frac{d\psi}{d\mu} = 0$ is $\mu = \cdot 8302$, corresponding to $\theta = 33^\circ\, 53' = \cdot 59137$ radians[1].

If we take $c\theta = r$, so that r is the radius of the small circle measured along the sphere, we get

$$\kappa r = \sqrt{(42)} \times \cdot 59137 = 3 \cdot 8325,$$

which is the equation connecting the value of κ $(= 2\pi\lambda^{-1})$ with the curved radius r, in the case of a small circle, whose angular radius is $33^\circ\, 53'$. If the layer were plane (§ 339), the value of κr would be $3 \cdot 8317$; so that it makes no perceptible difference in the pitch of the gravest tone whether the radius (r) of given length be

[1] The radian is the unit of circular measure.

straight, or be curved to an arc of 33°. The result of the comparison would, however, be materially different, if we were to take the length of the circumference as the same in the two cases, that is, replace $c\theta = r$ by $c\nu = r$.

In order to deduce the symmetrical solution for a plane layer, it is only necessary to make c infinite, while $c\nu$ remains finite. On account of the infinite value of h^2, the solution assumes the simple form.

$$\psi = A\left\{1 - \frac{h^2\nu^2}{2^2} + \frac{h^4\nu^4}{2^2 . 4^2} - \frac{h^6\nu^6}{2^2 . 4^2 . 6^2} + \dots\right\} \dots\dots\dots (4),$$

or, if we write $c\nu = r$, where r is the polar radius in two dimensions,

$$\psi = A\left\{1 - \frac{\kappa^2 r^2}{2^2} + \frac{\kappa^4 r^4}{2^2 . 4^2} - \dots\dots\dots\right\} = A\, J_0(\kappa r) \dots\dots\dots (5),$$

as in (14) § 336.

The differential equation for ψ in terms of ν, when c is infinite and $c\nu = r$, becomes

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \kappa^2\psi = 0 \dots\dots\dots\dots\dots\dots\dots\dots (6).$$

An independent investigation and solution for the plane problem will be given presently.

338. When s is different from zero, the differential equation satisfied by the coefficients of $\sin s\omega$, $\cos s\omega$, is

$$\nu^2(1 - \nu^2)\frac{d^2\psi_s}{d\nu^2} + \nu(1 - 2\nu^2)\frac{d\psi_s}{d\nu} + \nu^2 h^2\psi_s - s^2\psi_s = 0 \dots\dots (1),$$

and the solution, subject to the condition of finiteness when $\nu = 0$[1], is easily found to be

$$\psi_s = A\nu^s\left\{1 + \frac{s(s+1) - h^2}{2(2s+2)}\nu^2 + \frac{s(s+1) - h^2}{2(2s+2)} \cdot \frac{(s+2)(s+3) - h^2}{4(2s+4)}\nu^4 + \dots\right\};$$

or, if we put $h^2 = n(n+1)$,

$$\psi_s = A\nu^s\left\{1 + \frac{(s-n)(s+n+1)}{2 . (2s+2)}\nu^2 + \frac{(s-n)(s-n+2)(s+n+1)(s+n+3)}{2 . 4 . (2s+2)(2s+4)}\nu^4 + \dots\right\} \dots\dots (2).$$

[1] The solution may be completed by the addition of a second function derived from (2) by changing the sign of s, which occurs in (1) only as s^2, but a modification is necessary, when s is a positive integer. The method of procedure will be exemplified presently in the case of the plane layer.

We have here the complete solution of the problem of the vibrations of a spherical layer of gas bounded by a small circle whose radius is less than the quadrant. For each value of s, there are a series of possible values of n, determined by the condition $\dfrac{d\psi_s}{d\nu}=0$; with any of these values of n the function on the right-hand side of (2), when multiplied by $\cos s\omega$ or $\sin s\omega$, is a normal function of the system. The aggregate of all the normal functions corresponding to every admissible value of s and n, with an arbitrary coefficient prefixed to each, gives an expression capable of being identified with the initial value of ψ, i.e. with a function given arbitrarily over the area of the small circle.

When the radius of the sphere c is infinitely great, h^2 is infinite. If $c\nu=r$, $h^2\nu^2=\kappa^2r^2$, and (2) becomes

$$\psi_s=A r^s\left\{1-\frac{\kappa^2r^2}{2\,.\,(2s+2)}+\frac{\kappa^4r^4}{2\,.\,4\,.\,(2s+2)(2s+4)}-\dots\right\}\dots\dots(3),$$

a function of r proportional to $J_s(\kappa r)$.

In terms of μ, the differential equation satisfied by the coefficient of $\cos s\omega$, or $\sin s\omega$, is

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\psi_s}{d\mu}\right\}+h^2\psi_s-\frac{s^2}{1-\mu^2}\psi_s=0\dots\dots\dots\dots(4).$$

Assuming $\psi_s=(1-\mu^2)^{\frac{1}{2}s}\phi_s$, we find as the equation for ϕ_s

$$(1-\mu^2)\frac{d^2\phi_s}{d\mu^2}-2(s+1)\mu\frac{d\phi_s}{d\mu}+\{h^2-s(s+1)\}\phi_s=0\dots\dots(5),$$

which will be more easily dealt with.

To solve it, let

$$\phi_s=\mu^a+a_2\mu^{a+2}+a_4\mu^{a+4}+\dots+a_{2m}\mu^{a+2m}+\dots,$$

and substitute in (5). The coefficient of the lowest power of μ is $a(a-1)$; so that $a=0$, or $a=1$. The relation between a_{2m+2}, and a_{2m}, found by equating to zero the coefficient of μ^{a+2m}, is

$$a_{2m+2}=a_{2m}\frac{(a+2m+s-n)(a+2m+s+n+1)}{(a+2m+1)(a+2m+2)},$$

where $n(n+1)=h^2$.

The complete value of ϕ_s is accordingly given by

$$\phi_s = A\left\{1+\frac{(s-n)(s+n+1)}{1.2}\mu^2+\frac{(s-n)(s-n+2)(s+n+1)(s+n+3)}{1.2.3.4}\mu^4\right.$$
$$\left.+\frac{(s-n)(s-n+2)(s-n+4)(s+n+1)(s+n+3)(s+n+5)}{1.2.3.4.5.6}\mu^6+\ldots\right\}$$
$$+B\left\{\mu+\frac{(s-n+1)(s+n+2)}{2.3}\mu^3\right.$$
$$\left.+\frac{(s-n+1)(s-n+3)(s+n+2)(s+n+4)}{2.3.4.5}\mu^5+\ldots\right\}\ldots\ldots(6),$$

where A and B are arbitrary constants;

and $$\psi_s=(1-\mu^2)^{\frac{1}{2}s}\phi_s\ldots\ldots\ldots\ldots(7).$$

We have now to prove that the condition that neither pole is a source requires that $n-s$ be a positive integer, in which case one or other of the series in the expression for ϕ_s terminates. For this purpose it will not be enough to shew that the series (unless terminating) are infinite when $\mu=\pm 1$; it will be necessary to prove that they remain divergent after multiplication by $(1-\mu^2)^{\frac{1}{2}s}$, or as we may put it more conveniently, that they are infinite when $\mu=\pm1$ *in comparison with* $(1-\mu^2)^{-\frac{1}{2}s}$. It will be sufficient to consider in detail the case of the first series.

We have

$$1+\frac{(s-n)(s+n+1)}{1.2}+\frac{(s-n)(s-n+2)(s+n+1)(s+n+3)}{1.2.3.4}+\ldots\ldots$$
$$=1+\frac{(\frac{1}{2}s-\frac{1}{2}n)(\frac{1}{2}s+\frac{1}{2}n+\frac{1}{2})}{1.\frac{1}{2}}$$
$$+\frac{(\frac{1}{2}s-\frac{1}{2}n)(\frac{1}{2}s-\frac{1}{2}n+1)(\frac{1}{2}s+\frac{1}{2}n+\frac{1}{2})(\frac{1}{2}s+\frac{1}{2}n+\frac{1}{2}+1)}{1.2.\frac{1}{2}.\frac{3}{2}}+\ldots;$$

which is of the standard form (8) § 336

$$1+\frac{ab}{cd}+\frac{a(a+1)b(b+1)}{c(c+1)d(d+1)}+\ldots,$$

if $\quad a=\frac{1}{2}s-\frac{1}{2}n,\quad b=\frac{1}{2}s+\frac{1}{2}n+\frac{1}{2},\quad c=1,\quad d=\frac{1}{2}.$

The degree of divergency is determined by the value of $a+b-c-d$, which is here equal to $s-1$.

On the other hand, the binomial theorem gives for the expansion of $(1-\mu^2)^{-\frac{1}{2}s}$

$$1 + \frac{\frac{1}{2}s}{1}\mu^2 + \frac{\frac{1}{2}s(\frac{1}{2}s+1)}{1\,.\,2}\mu^4 + \ldots\ldots,$$

which is of the standard form, if

$$a = \tfrac{1}{2}s, \quad c = 1, \quad b = d, \quad \text{and makes} \quad a + b - c - d = \tfrac{1}{2}s - 1.$$

Since $s - 1 > \frac{1}{2}s - 1$, it appears that the series in the expression for ϕ_s are infinities of a higher order than $(1-\mu^2)^{-\frac{1}{2}s}$, and therefore remain infinite after multiplication by $(1-\mu^2)^{\frac{1}{2}s}$. Accordingly ψ_s cannot be finite at both poles unless one or other of the series terminate, which can only happen when $n - s$ is zero, or a positive integer. If the integer be even, we have still to suppose $B = 0$; and if the integer be odd, $A = 0$, in order to secure finiteness at the poles.

In either case the value of ϕ_s for the complete sphere may be put into the form

$$\phi_s = \frac{d^{n+s}}{d\mu^{n+s}}(1-\mu^2)^n = \frac{d^s P_n(\mu)}{d\mu^s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

where the constant multiplier is omitted. The complete expression for that part of ψ which contains $\cos s\omega$ or $\sin s\omega$ as a factor is therefore

$$\psi = \frac{\cos s\omega}{\sin s\omega} \sum_{n=s}^{n=\infty} A_n \nu^s \frac{d^s}{d\mu^s} P_n(\mu) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

where A_n is constant with respect to μ and ω, but as a function of the time will vary as

$$\cos\left(\frac{\sqrt{(n\,.\,n+1)}\,at}{c} + \epsilon\right) \ldots\ldots\ldots\ldots\ldots\ldots(10).$$

For most purposes, however, it is more convenient to group the terms for which n is the same, rather than those for which s is the same. Thus for any value of n

$$\psi = \sum_{s=0}^{s=n} \nu^s \frac{d^s P_n(\mu)}{d\mu^s}(A_s \cos s\omega + B_s \sin s\omega) \ldots\ldots\ldots(11),$$

where every coefficient A_s, B_s may be regarded as containing a time factor of the form (10).

Initially ψ is an arbitrary function of μ and ω, and therefore any such function is capable of being represented in the form

$$\psi = \sum_{n=0}^{n=\infty} \sum_{s=0}^{s=n} \nu^s \frac{d^s P_n(\mu)}{d\mu^s} (A_s^n \cos s\omega + B_s^n \sin s\omega) \ldots\ldots (12),$$

which is Laplace's expansion in spherical surface harmonics.

From the differential equation (5), or from its general solution (6), it is easy to prove that ϕ_s is of the same form as $\frac{d}{d\mu}\phi_{s-1}$, so that we may write

$$\phi_s = \left(\frac{d}{d\mu}\right)^s \phi_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

(in which no connection between the arbitrary constants is asserted), or in terms of ψ by (7),

$$\psi_s = (1-\mu^2)^{\frac{1}{2}s} \left(\frac{d}{d\mu}\right)^s \psi_0 \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

Equation (13) is a generalization of the property of Laplace's functions used in (8).

The corresponding relations for the plane problem may be deduced, as before, by attaching an infinite value to n, which in (13), (14) is arbitrary, and writing $n\nu = \kappa r$. Since $\mu^2 + \nu^2 = 1$,

$$\psi_s = \nu^s \left(-\frac{\mu}{\nu}\frac{d}{d\nu}\right)^s \psi_0,$$

ψ_0 being regarded as a function of ν. In the limit μ (even though subject to differentiation) may be identified with unity, and thus we may take

$$\psi_s = (-2\kappa r)^s \left(\frac{d}{d\,.\,(\kappa r)^2}\right)^s \psi_0 \ldots\ldots\ldots\ldots\ldots (15).$$

When the pole is not a source, ψ_s is proportional to $J_s(\kappa r)$. The constant coefficient, left undetermined by (15), may be readily found by a comparison of the leading terms. It thus appears that

$$J_s(\kappa r) = (-2\kappa r)^s \left(\frac{d}{d\,.\,(\kappa r)^2}\right)^s J_0(\kappa r) \ldots\ldots\ldots\ldots (16),$$

a well-known property of Bessel's functions[1].

The vibrations of a plane layer of gas are of course more easily dealt with, than those of a layer of finite curvature, but

[1] Todhunter's *Laplace's Functions*, § 390.

I have preferred to exhibit the indirect as well as the direct method of investigation, both for the sake of the spherical problem itself with the corresponding Laplace's expansion[1], and because the connection between Bessel's and Laplace's functions appears not to be generally understood. We may now, however, proceed to the independent treatment of the plane problem.

339. If in the general equation of simple aerial vibrations

$$\nabla^2\psi + \kappa^2\psi = 0,$$

we assume that ψ is independent of z, and introduce plane polar coordinates, we get (§ 241)

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} + \kappa^2\psi = 0 \text{ ....(1)};$$

or, if ψ be expanded in Fourier's series

$$\psi = \psi_0 + \psi_1 + \dots + \psi_n + \text{..................(2)},$$

where ψ_n is of the form $A_n \cos n\theta + B_n \sin n\theta$,

$$\frac{d^2\psi_n}{dr^2} + \frac{1}{r}\frac{d\psi_n}{dr} + \left(\kappa^2 - \frac{n^2}{r^2}\right)\psi_n = 0 \text{ ....(3)}[2].$$

This equation is of the same form as that with which we had to deal in treating of circular membranes (§ 200); the principal mathematical difference between the two questions lies in the fact that while in the case of membranes the condition to be satisfied at the boundary is $\psi = 0$, in the present case interest attaches itself rather to the boundary condition $\frac{d\psi}{dr} = 0$, corresponding to the confinement of the gas by a rigid cylindrical envelope.

The pole not being a source, the solution of (3) is

$$\psi_n = A\, J_n(\kappa r) \text{ ....(4)},$$

and the equation giving the possible periods of vibration within a cylinder of radius r, is

$$J_n'(\kappa r) = 0 \text{..........................(5)}.$$

[1] I have been much assisted by Heine's *Handbuch der Kugelfunctionen*, Berlin, 1861, and by Sir W. Thomson's papers on Laplace's Theory of the Tides, *Phil. Mag.* Vol. IV. 1875.

[2] I here recur to the usual notation, but the reader will understand that n corresponds to the s of preceding sections. The n of Laplace's functions is now infinite.

The lower values of κr satisfying (5) are given in the following table[1], which was calculated from Hansen's tables of the functions J by means of the relations allowing J_n to be expressed in terms of J_0 and J_1.

Number of internal circular nodes.	$n=0$	$n=1$	$n=2$	$n=3$
0	3·832	1·841	3·054	4·201
1	7·015	5·332	6·705	8·015
2	10·174	8·536	9·965	11·344
3	13·324	11·706		
4	16·471	14·864		
5	19·616	18·016		

The particular solution may be written

$$\psi_n = (A\cos n\theta + B\sin n\theta)\, J_n(\kappa r)\cos\kappa at$$
$$+ (C\cos n\theta + D\sin n\theta)\, J_n(\kappa r)\sin\kappa at \quad\text{.............(6)},$$

where A, B, C, D are arbitrary for every admissible value of n and κ. As in the corresponding problems for the sphere and circular membrane, the sum of all the particular solutions must be general enough to represent, when $t=0$, arbitrary values of ψ and $\dot{\psi}$.

As an example of compound vibrations we may suppose, as in § 332, that the initial condition of the gas is that defined by

$$\dot{\psi}=0, \quad \psi = x = r\cos\theta.$$

Under these circumstances (6) reduces to

$$\psi = A_1\cos\theta\, J_1(\kappa_1 r)\cos\kappa_1 at + A_2\cos\theta\, J_1(\kappa_2 r)\cos\kappa_2 at + \ldots (7),$$

and, if we suppose the radius of the cylinder to be unity, the admissible values of κ are the roots of

$$J_1'(\kappa) = 0 \quad\text{..........................(8)}.$$

The condition to determine the coefficients A is that for all values of r from $r=0$ to $r=1$,

$$r = A_1 J_1(\kappa_1 r) + A_2 J_1(\kappa_2 r) + \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

[1] Notes on Bessel's Functions. *Phil. Mag.* Nov. 1872.

whence, as in § 332,

$$A = \frac{2}{(\kappa^2 - 1)\, J_1(\kappa)} \quad \text{..................}(10).$$

The complete solution is therefore

$$\psi = \Sigma \frac{2 \cos\theta \, J_1(\kappa r)}{(\kappa^2 - 1)\, J_1(\kappa)} \cos \kappa a t \quad \text{..............}(11),$$

where the summation extends to all the values of κ determined by (8).

If we put $t = 0$ and $r = 1$, we get from (9) and (10)

$$\Sigma \frac{2}{\kappa^2 - 1} = 1 \quad \text{......................}(12),$$

an equation which may be verified numerically, or by an analytical process similar to that applied in the case of (14) § 332. We may prove that

$$\log J_1'(z) = \text{constant} + \Sigma \log\left(1 - \frac{z^2}{\kappa^2}\right),$$

whence by differentiation

$$\frac{J_1''(z)}{J_1'(z)} = -\Sigma \frac{2z}{\kappa^2 - z^2}.$$

From this (12) is derived by putting $z = 1$, and having regard to the fundamental differential equation satisfied by J_1, which shews that

$$J_1''(1) : J_1'(1) = -1.$$

Hitherto we have supposed the cylinder complete, so that ψ recurs after each revolution, which requires that n be integral; but if instead of the complete cylinder we take the sector included between $\theta = 0$ and $\theta = \beta$, fractional values of n will in general present themselves. Since $\frac{d\psi}{d\theta}$ vanishes at both limits of θ, ψ must be of the form

$$\psi = A \cos(\kappa a t + \epsilon) \cos n\theta \, J_n(\kappa r) \quad \text{...........}(13),$$

where $n = \nu\pi\beta^{-1}$, ν being integral. If β be an aliquot part of π (or π itself), the complete solution involves only integral values of n, as might have been foreseen; but, in general, functions of fractional order must be introduced.

An interesting example occurs when $\beta = 2\pi$, which corresponds to the case of a cylinder, traversed by a rigid wall

stretching from the centre to the circumference (compare § 207). The effect of the wall is to render possible a difference of pressure on its two sides; but when no such difference occurs, the wall may be removed, and the vibrations are included under the theory of a complete cylinder. This state of things occurs when ν is even. But when ν is odd, n is of the form (integer $+\frac{1}{2}$), and the pressures on the two sides of the wall are different. In the latter case J_n is expressible in finite terms. The gravest tone is obtained by taking $\nu=1$, or $n=\frac{1}{2}$, when

$$\psi = A \cos(\kappa at + \epsilon) \cdot \cos \tfrac{1}{2}\theta \cdot \frac{\sin \kappa r}{\sqrt{(\kappa r)}} \quad \ldots\ldots\ldots\ldots(14),$$

and the admissible values of κ are the roots of $\tan \kappa = 2\kappa$. The first root (after $\kappa=0$) is $\kappa = 1{\cdot}1655$, corresponding to a tone decidedly graver than any one, of which the complete cylinder is capable.

The preceding analysis has an interesting application to the mathematically analogous problem of the vibrations of water in a cylindrical vessel of uniform depth. The reader may consult a paper on waves by the author in the *Philosophical Magazine* for April, 1876, and papers by Prof. Guthrie to which reference is there made. The observation of the periodic time is very easy, and in this way may be obtained an experimental solution of problems, whose theoretical treatment is far beyond the power of known methods.

340. Returning to the complete cylinder, let us suppose it closed by rigid transverse walls at $z=0$, and $z=l$, and remove the restriction that the motion is to be the same in all transverse sections. The general differential equation (§ 241) is

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} + \frac{d^2\psi}{dz^2} + \kappa^2\psi = 0 \quad \ldots\ldots\ldots (1).$$

Let ψ be expanded by Fourier's theorem in the series

$$\psi = H_0 + H_1 \cos\frac{\pi z}{l} + H_2 \cos\frac{2\pi z}{l} + \ldots + H_p \cos\left(p\frac{\pi z}{l}\right) + \ldots \quad (2),$$

where the coefficients H_p may be functions of r and θ. This form secures the fulfilment of the boundary conditions, when $z=0$, $z=l$,

and each term must satisfy the differential equation separately. Thus

$$\frac{d^2 H_p}{dr^2} + \frac{1}{r}\frac{dH_p}{dr} + \frac{1}{r^2}\frac{d^2 H_p}{d\theta^2} + \left(\kappa^2 - p^2\frac{\pi^2}{l^2}\right) H_p = 0 \text{ ....(3)},$$

which is of the same form as when the motion is independent of z, κ^2 being replaced by $\kappa^2 - p^2\pi^2 l^{-2}$. The particular solution may therefore be written

$$\psi = (A_n \cos n\theta + B_n \sin n\theta) \,.\, \cos p\frac{\pi z}{l} \,.\, J_n(\sqrt{\kappa^2 - p^2\pi^2 l^{-2}} \,.\, r) \cos \kappa a t$$

$$+ (C_n \cos n\theta + D_n \sin n\theta) \cos p\frac{\pi z}{l} \,.\, J_n(\sqrt{\kappa^2 - p^2\pi^2 l^{-2}} \,.\, r) \sin \kappa a t \text{...(4)},$$

which must be generalized by a triple summation, with respect to all integral values of p and n, and also with respect to all the values of κ, determined by the equation,

$$J_n'(\sqrt{\kappa^2 - p^2\pi^2 l^{-2}} \,.\, r) = 0 \text{ ....(5)}.$$

If $r = 1$, and K denote the values of κ given in the table (§ 339), corresponding to purely transverse vibrations, we have

$$\kappa^2 = K^2 + p^2\frac{\pi^2}{l^2} \text{ ....(6)}.$$

The purely axial vibrations correspond to a zero value of K, not included in the table.

341. The complete integral of the equation

$$\frac{d^2\psi_n}{dr^2} + \frac{1}{r}\frac{d\psi_n}{dr} + \left(\kappa^2 - \frac{n^2}{r^2}\right)\psi_n = 0 \text{ ....(1)},$$

when there is no limitation as to the absence of a source at the pole, involves a second function of r, which may be denoted by $J_{-n}(\kappa r)$. Thus, omitting unnecessary constant multipliers, we may take (§ 200)

$$\psi_n = A r^{+n}\left\{1 - \frac{\kappa^2 r^2}{2 \,.\, 2+2n} + \frac{\kappa^4 r^4}{2 \,.\, 4 \,.\, 2+2n \,.\, 4+2n} - \ldots\right\}$$

$$+ B r^{-n}\left\{1 - \frac{\kappa^2 r^2}{2 \,.\, 2-2n} + \frac{\kappa^4 r^4}{2 \,.\, 4 \,.\, 2-2n \,.\, 4-2n} - \ldots\right\} \text{.......(2)},$$

but the second series requires modification, if n be integral. When $n = 0$, the two series become identical, and thus the immediate result of supposing $n = 0$ in (2) lacks the necessary generality. The

required solution may, however, be obtained by the ordinary rule applicable to such cases. Denoting the coefficients of A and B in (2) by $f(n)$, $f(-n)$, we have

$$\begin{aligned}\psi &= Af(n) + Bf(-n)\\ &= (A+B)f(0) + (A-B)f'(0)\,n + (A+B)f''(0)\frac{n^2}{1.2} + \ldots\end{aligned}$$

by Maclaurin's theorem. Hence, taking new arbitrary constants, we may write as the limiting form of (2),

$$\psi_0 = Af(0) + Bf'(0).$$

In this equation $f(0)$ is $J_0(\kappa r)$; to find $f'(0)$ we have

$$\begin{aligned}f'(n) &= r^n \log r\left\{1 - \frac{\kappa^2 r^2}{2.2+2n} + \frac{\kappa^4 r^4}{2.4.2+2n.4+2n} - \ldots\right\}\\ &\quad + r^n \frac{d}{dn}\left\{1 - \frac{\kappa^2 r^2}{2.2+2n} + \frac{\kappa^4 r^4}{2.4.2+2n.4+2n} - \ldots\right\}.\end{aligned}$$

If u denote the general term (involving r^{2m}) of the series within brackets, taken without regard to sign,

$$\frac{1}{u}\frac{du}{dn} = \frac{d\log u}{dn} = -\frac{2}{2+2n} - \frac{2}{4+2n} - \ldots - \frac{2}{2m+2n},$$

so that $$\left(\frac{du}{dn}\right)_{n=0} = -u_{n=0}S_m,$$

if $$S_m = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m} \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Thus $$\begin{aligned}f'(0) &= \log r\left\{1 - \frac{\kappa^2 r^2}{2^2} + \frac{\kappa^4 r^4}{2^2.4^2} - \frac{\kappa^6 r^6}{2^2.4^2.6^2} + \ldots\right\}\\ &\quad + \left\{\frac{\kappa^2 r^2}{2^2} - \frac{\kappa^4 r^4}{2^2.4^2}S_2 + \frac{\kappa^6 r^6}{2^2.4^2.6^2}S_3 - \ldots\right\},\end{aligned}$$

and the complete integral for the case $n=0$ is

$$\begin{aligned}\psi_0 &= (A + B\log r)\left\{1 - \frac{\kappa^2 r^2}{2^2} + \frac{\kappa^4 r^4}{2^2.4^2} - \ldots\right\}\\ &\quad + B\left\{\frac{\kappa^2 r^2}{2^2} - \frac{\kappa^4 r^4}{2^2.4^2}S_2 + \frac{\kappa^6 r^6}{2^2.4^2.6^2}S_3 - \ldots\right\}\ldots\ldots\ldots\ldots(4).\end{aligned}$$

For the general integral value of n, the corresponding expression may be derived by means of (15) § 338

$$\psi_n = (-2\kappa r)^n\left(\frac{d}{d(\kappa r)^2}\right)^n\psi_0 \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The formula of derivation (5) may be obtained directly from the differential equation (1). Writing z for κr and putting

$$\psi_n = z^n \phi_n \quad\text{.............................. (6),}$$

we find in place of (1)

$$\frac{d^2\phi_n}{dz^2} + \frac{2n+1}{z}\frac{d\phi_n}{dz} + \phi_n = 0 \quad\text{................ (7).}$$

Again (7) may be put into the form

$$z^2\frac{d^2\phi_n}{d(z^2)^2} + (n+1)\frac{d\phi_n}{d\,.\,z^2} + \tfrac{1}{4}\phi_n = 0 \quad\text{.............. (8),}$$

from which it follows at once that

$$\phi_n = \frac{d}{d\,.\,z^2}\phi_{n-1} \quad\text{........................ (9);}$$

so that

$$\phi_n = \left(\frac{d}{d\,.\,z^2}\right)^n \phi_0 \quad\text{...................... (10),}$$

or by (6)

$$\psi_n = z^n \left(\frac{d}{d\,.\,z^2}\right)^n \psi_0 \quad\text{.................... (11),}$$

which is equivalent to (5), since the constants in ψ_0 are arbitrary in both equations.

The serial expressions for ψ_n thus obtained are convergent for all values of the argument, but are practically useless when the argument is great. In such cases we must have recourse to semi-convergent series corresponding to that of (10) § 200.

Equation (1) may be put into the form

$$\frac{d^2(z^{\frac12}\psi_n)}{dz^2} - \frac{(n-\frac12)(n+\frac12)}{z^2}(z^{\frac12}\psi_n) + z^{\frac12}\psi_n = 0 \quad\text{......... (12),}$$

whence by § 323 (4), (12), we find as the general solution of (1)

$$\psi_n = C(i\kappa r)^{-\frac12}e^{-i\kappa r}\left\{1 - \frac{1^2-4n^2}{1\,.\,8i\kappa r} + \frac{(1^2-4n^2)(3^2-4n^2)}{1\,.\,2\,(8i\kappa r)^2} - \frac{(1^2-4n^2)(3^2-4n^2)(5^2-4n^2)}{1\,.\,2\,.\,3\,.\,(8i\kappa r)^3} + \ldots\right\}$$

$$+ D(i\kappa r)^{-\frac12}e^{+i\kappa r}\left\{1 + \frac{1^2-4n^2}{1\,.\,8i\kappa r} + \frac{(1^2-4n^2)(3^2-4n^2)}{1\,.\,2\,.\,(8i\kappa r)^2} + \frac{(1^2-4n^2)(3^2-4n^2)(5^2-4n^2)}{1\,.\,2\,.\,3\,.\,(8i\kappa r)^3} + \ldots\right\} \quad\text{.......... (13).}$$

When n is integral, these series are infinite and ultimately divergent, but (§§ 200, 302) this circumstance does not interfere with their practical utility.

The most important application of the complete integral of (1) is to represent a disturbance diverging from the pole, a problem which has been treated by Stokes in his memoir on the communication of vibrations to a gas. The condition that the disturbance represented by (13) shall be exclusively divergent is simply $D=0$, as appears immediately on introduction of the time factor e^{ikat} by supposing r to be very great; the principal difficulty of the question consists in discovering what relation between the coefficients of the ascending series corresponds to this condition, for which purpose Stokes employs the solution of (1) in the form of a definite integral. We shall attain the same object, perhaps more simply, by using the results of § 302.

By (22), (24) § 302

$$-\left(\frac{\pi}{2iz}\right)^{\frac{1}{2}} e^{-iz}\left\{1-\frac{1^2}{1.8iz}+\frac{1^2.3^2}{1.2.(8iz)^2}-\ldots\right\}$$
$$=\tfrac{1}{2}\pi\{K(z)+iJ_0(z)\}-\int_0^\infty \frac{e^{-\beta}d\beta}{\sqrt{(\beta^2+z^2)}}\quad\ldots\ldots\ldots(14),$$

and thus the question reduces itself to the determination of the form of the right-hand member of (14) when z is small. By (5) § 302 and (5) § 200 we have

$$\tfrac{1}{2}\pi\{K(z)+iJ_0(z)\}=z+\tfrac{1}{2}i\pi+\text{higher terms in } z\ \ldots\ldots(15),$$

so that all that remains is to find the form of the definite integral in (14), when z is small. Putting $\sqrt{(\beta^2+z^2)}=y-\beta$, we have

$$\int_0^\infty \frac{e^{-\beta}d\beta}{\sqrt{(\beta^2+z^2)}}=\int_z^\infty e^{-\frac{y^2-z^2}{2y}}\frac{dy}{y}=\int_z^\infty e^{\frac{1}{2}z^2y^{-1}}e^{-\frac{1}{2}y}\frac{dy}{y}.$$

When z is small, $z^2(2y)^{-1}$ is also small throughout the range of integration, and thus we may write

$$\int_0^\infty \frac{e^{-\beta}d\beta}{\sqrt{(\beta^2+z^2)}}=\int_z^\infty\left\{1+\frac{z^2}{2y}+\frac{1}{2}\frac{z^4}{4y^2}+\ldots\right\}\frac{e^{-\frac{1}{2}y}}{y}dy.$$

The first integral on the right is

$$\int_z^\infty \frac{e^{-\frac{1}{2}y}}{y}dy=\int_{\frac{1}{2}z}^\infty \frac{e^{-v}dv}{v}=-\gamma-\log(\tfrac{1}{2}z)+\tfrac{1}{2}z+\ldots\ldots\ (16)^1,$$

[1] De Morgan's *Differential and Integral Calculus*, p. 653.

where γ is Euler's constant (·5772...); and, as we may easily satisfy ourselves by integration by parts, the other integrals do not contribute anything to the leading terms. Thus, when z is very small,

$$-\left(\frac{\pi}{2iz}\right)^{\frac{1}{2}} e^{-iz}\left\{1-\frac{1^2}{1.8iz}+\frac{1^2.3^2}{1.2.(8iz)^2}-\ldots\right\}$$
$$=\gamma+\log\left(\tfrac{1}{2}z\right)+\tfrac{1}{2}i\pi+\ldots\ldots\ldots \quad (17).$$

Replacing z by κr, and comparing with the form assumed by (4), when r is small, we see that in order to make the series identical we must take

$$A=\gamma+\log\tfrac{1}{2}+\log\kappa+\tfrac{1}{2}i\pi, \qquad B=1;$$

so that a series of waves diverging from the pole, whose expression in descending series is

$$\psi_0=-\left(\frac{\pi}{2i\kappa r}\right)^{\frac{1}{2}} e^{-i\kappa r}\left\{1-\frac{1^2}{1.8i\kappa r}+\frac{1^2.3^2}{1.2.(8i\kappa r)^2}-\ldots\right\}\ldots\ldots \quad (18),$$

is represented also by the ascending series

$$\psi_0=\left(\gamma+\log\frac{i\kappa r}{2}\right)\left\{1-\frac{\kappa^2r^2}{2^2}+\frac{\kappa^4r^4}{2^2.4^2}-\ldots\right\}$$
$$+\frac{\kappa^2r^2}{2^2}S_1-\frac{\kappa^4r^4}{2^2.4^2}S_2+\frac{\kappa^6r^6}{2^2.4^2.6^2}S_3-\ldots\ldots\ldots \quad (19).$$

In applying the formula of derivation (11) to the descending series, the parts containing $e^{-i\kappa r}$ and $e^{+i\kappa r}$ as factors will evidently remain distinct, and the complete integral for the general value of n, subject to the condition that the part containing $e^{+i\kappa r}$ shall not appear, will be got by differentiation from the complete integral for $n=0$ subject to the same condition. Thus, since by (5) $\psi_1=\dfrac{d\psi_0}{dr}$,

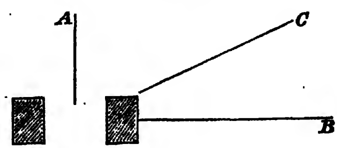
$$\psi_1=\left(\frac{\pi i}{2\kappa r}\right)^{\frac{1}{2}} e^{-i\kappa r}\left\{1-\frac{-1.3}{1.8i\kappa r}+\frac{-1.1.3.5}{1.2.(8i\kappa r)^2}\right.$$
$$\left.-\frac{-1.1.3^2.5.7}{1.2.3.(8i\kappa r)^3}+\ldots\right\}\ldots\ldots\ldots \quad (20),$$

or, in terms of the ascending series,

$$\psi_1 = \frac{1}{\kappa r}\left\{1 - \frac{\kappa^2 r^2}{2^2} + \frac{\kappa^4 r^4}{2^2 . 4^2} - \ldots\right\}$$
$$- \left(\gamma + \log \frac{i\kappa r}{2}\right)\left\{\frac{\kappa r}{2} - \frac{\kappa^3 r^3}{2^2 . 4} + \frac{\kappa^5 r^5}{2^2 . 4^2 . 6} - \ldots\right\}$$
$$+ \frac{\kappa r}{2} S_1 - \frac{\kappa^3 r^3}{2^2 . 4} S_2 + \frac{\kappa^5 r^5}{2^2 . 4^2 . 6} S_3 - \ldots\ldots\ldots \quad (21).$$

These expressions are applied by Prof. Stokes to shew how feebly the vibrations of a string, (corresponding to the term of order one), are communicated to the surrounding gas. For this purpose he makes a comparison between the actual sound, and what would have been emitted in the same direction, were the lateral motion of the gas in the neighbourhood of the string prevented. For a piano string corresponding to the middle C, the radius of the wire may be about ·02 inch, and λ is about 25 inches; and it appears that the sound is nearly 40,000 times weaker than it would have been if the motion of the particles of air had taken place in planes passing through the axis of the string. "This shews the vital importance of sounding-boards in stringed instruments. Although the amplitude of vibration of the particles of the sounding-board is extremely small compared with that of the particles of the string, yet as it presents a broad surface to the air it is able to excite loud sonorous vibrations, whereas were the string supported in an absolutely rigid manner, the vibrations which it could excite directly in the air would be so small as to be almost or altogether inaudible."

Fig. 64.



"The increase of sound produced by the stoppage of lateral motion may be prettily exhibited by a very simple experiment. Take a tuning-fork, and holding it in the fingers after it has been

made to vibrate, place a sheet of paper, or the blade of a broad knife, with its edge parallel to the axis of the fork, and as near to the fork as conveniently may be without touching. If the plane of the obstacle coincide with either of the planes of symmetry of the fork, as represented in section at A or B, no effect is produced; but if it be placed in an intermediate position, such as C, the sound becomes much stronger[1]."

342. The real expression for the velocity-potential of symmetrical waves diverging in two dimensions is obtained from (18) § 341 after introduction of the time factor $e^{i\kappa at}$ by rejecting the imaginary part; it is

$$\psi_0 = -\left(\frac{\pi}{2\kappa r}\right)^{\frac{1}{2}} \cos\kappa\,(at - r - \tfrac{1}{8}\lambda)\left\{1 - \frac{1^2 \cdot 3^2}{1 \cdot 2 \cdot (8\kappa r)^2} + \dots\right\}$$

$$+ \left(\frac{\pi}{2\kappa r}\right)^{\frac{1}{2}} \sin\kappa\,(at - r - \tfrac{1}{8}\lambda)\left\{\frac{1^2}{1 \cdot 8\kappa r} - \frac{1^2 \cdot 3^2 \cdot 5^2}{1 \cdot 2 \cdot 3 \cdot (8\kappa r)^3} + \dots\right\} \dots\dots (1),$$

in which, as usual, two arbitrary constants may be inserted, one as a multiplier of the whole expression and the other as an addition to the time.

The problem of a linear source of uniform intensity may also be treated by the general method applicable in three dimensions. Thus by (3) § 277, if ρ be the distance of any element dx from O, the point at which the potential is to be estimated, and r be the smallest value of ρ, so that $\rho^2 = r^2 + x^2$, we may take

$$\phi = 2\int_0^\infty \frac{e^{-i\kappa\rho}\,dx}{\rho} = 2\int_r^\infty \frac{e^{-i\kappa\rho}\,d\rho}{\sqrt{(\rho^2 - r^2)}} \dots\dots\dots\dots (2),$$

which must be of the same form as (1). Taking $y = \rho - r$, we may write in place of (2)

$$\phi = 2\int_0^\infty \frac{e^{-i\kappa r} e^{-i\kappa y}\,dy}{\sqrt{y} \cdot \sqrt{(2r + y)}} \dots\dots\dots\dots\dots\dots\dots (3),$$

from which the various expressions follow as in (14) § 341. When κr is great, an approximate value of the integral may be obtained by neglecting the variation of $\sqrt{(2r + y)}$, since on account of the rapid fluctuation of sign caused by the factor $e^{-i\kappa y}$ we need attend

[1] *Phil. Trans.* 1868.

only to small values of y. Now

$$\int_0^\infty \frac{\cos x\,dx}{\sqrt{x}} = \int_0^\infty \frac{\sin x\,dx}{\sqrt{x}} = \sqrt{\left(\frac{\pi}{2}\right)} \text{ .... } (4),$$

so that $$\phi = \sqrt{\left(\frac{\pi}{\kappa r}\right)} e^{-i\kappa r}(1-i) = \sqrt{\left(\frac{2\pi}{\kappa r}\right)} e^{-i\kappa(r+\frac{1}{8}\lambda)} \text{ .... } (5).$$

Introducing the factor $e^{i\kappa at}$, and rejecting the imaginary part of the expression, we have finally

$$\phi = \sqrt{\left(\frac{2\pi}{\kappa r}\right)} \cos\kappa\,(at - r - \tfrac{1}{8}\lambda) \text{ .... } (6),$$

as the value of the velocity-potential at a great distance. A similar argument is applicable to shew that (1) is also the expression for the velocity-potential on one side of an infinite plane (§ 278) due to the uniform normal motion of an infinitesimal strip bounded by parallel lines.

In like manner we may regard the term of the first order (20) § 341 as the expression of the velocity-potential due to double sources uniformly distributed along an infinite straight line.

From the point of view of the present section we see the significance of the retardation of $\frac{1}{8}\lambda$, which appears in (1) and in the results of the following section (16), (17). In the ordinary integration for surface distributions by Huyghens' zones (§ 283) the whole effect is the half of that of the first zone, and the phase of the effect of the first zone is midway between the phases due to its extreme parts, i.e. $\frac{1}{4}\lambda$ behind the phase due to the central point. In the present case the retardation of the resultant relatively to the central element is less, on account of the preponderance of the central parts.

343. In illustration of the formulæ of § 341 we may take the problem of the disturbance of plane waves of sound by a cylindrical obstacle, whose radius is small in comparison with the length of the waves, and whose axis is parallel to their plane. (Compare § 335.)

Let the plane waves be represented by

$$\phi = e^{i\kappa(at+x)} = e^{i\kappa at} \cdot e^{i\kappa r\cos\theta} \text{ .... } (1).$$

The general expansion of ϕ in Fourier's series may be readily effected, the coefficients of the various terms being, as might

be anticipated, simply the Bessel's functions of corresponding orders; but, as we confine ourselves here to the case where c the radius of the cylinder is small, we will at once expand in powers of r.

Thus, when $r=c$, if $e^{i\kappa at}$ be omitted,

$$\phi = 1 - \tfrac{1}{4}\kappa^2c^2 + i\kappa c\,.\cos\theta + \ldots\ldots\ldots\ldots\ldots(2),$$

$$\frac{d\phi}{dr} = -\tfrac{1}{2}\kappa^2 c + i\kappa\,.\cos\theta + \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The amount and even the law of the disturbance depends upon the character of the obstacle. We will begin by supposing the material of the cylinder to be a gas of density σ' and compressibility m'; the solution of the problem for a rigid obstacle may finally be derived by suitable suppositions with respect to σ', m'. If κ' be the internal value of κ, we have inside the cylinder by the condition that the axis is not a source (§ 339),

$$\psi = A_0\left\{1 - \frac{\kappa'^2r^2}{2^2} + \frac{\kappa'^4r^4}{2^2.4^2} - \ldots\right\} + A_1 r\left\{1 - \frac{\kappa'^2r^2}{2.4} + \frac{\kappa'^4r^4}{2.4^2.6} - \ldots\right\}\cos\theta;$$

so that, when $r=c$,

$$\psi\,(\text{inside}) = A_0\,(1 - \tfrac{1}{4}\kappa'^2c^2) + A_1 c\,(1 - \tfrac{1}{8}\kappa'^2c^2).\cos\theta\ldots\,(4),$$

$$\frac{d\psi}{dr}\,(\text{inside}) = -\tfrac{1}{2}A_0\,\kappa'^2 c + A_1\,(1 - \tfrac{3}{8}\kappa'^2c^2)\cos\theta\ldots\ldots\ldots(5).$$

Outside the cylinder, when $r=c$, we have by (19), (21) § 341,

$$\psi = B_0\left(\gamma + \log\frac{i\kappa c}{2}\right) + \frac{B_1\cos\theta}{\kappa c}\ldots\ldots\ldots\ldots(6),$$

$$\frac{d\psi}{dr} = \frac{B_0}{c} - \frac{B_1\cos\theta}{\kappa c^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

The conditions to be satisfied at the surface of separation are thus

$$-A_0\,\kappa'^2c^2 = -\kappa^2c^2 + 2B_0\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

$$\frac{\sigma'}{\sigma}A_0(1 - \tfrac{1}{4}\kappa'^2c^2) = 1 - \tfrac{1}{4}\kappa^2c^2 + B_0\left(\gamma + \log\frac{i\kappa c}{2}\right)\ldots..(9),$$

$$A_1\left(1 - \frac{3\kappa'^2c^2}{8}\right) = i\kappa - \frac{B_1}{\kappa c^2}\ldots\ldots\ldots\ldots\ldots(10),$$

$$\frac{\sigma'}{\sigma}A_1 c\left(1 - \frac{\kappa'^2c^2}{8}\right) = i\kappa c + \frac{B_1}{\kappa c}\ldots\ldots\ldots\ldots\ldots(11),$$

from which by eliminating A_0, A_1 we get approximately

$$B_0 = \tfrac{1}{2}\kappa^2c^2\left(1 - \frac{\kappa'^2}{\kappa^2}\cdot\frac{\sigma}{\sigma'}\right) = \tfrac{1}{2}\kappa^2c^2\frac{m'-m}{m'} \text{........(12)},$$

$$B_1 = i\kappa^2c^2\frac{\sigma'-\sigma}{\sigma'+\sigma} \text{....................(13)}.$$

Thus at a distance from the cylinder we have by (18) and (20) § 341,

$$\psi = -B_0\left(\frac{\pi}{2i\kappa r}\right)^{\frac{1}{2}} e^{-i\kappa r} + B_1\left(\frac{\pi i}{2\kappa r}\right)^{\frac{1}{2}} e^{-i\kappa r}\,.\cos\theta$$

$$= -\kappa^2c^2e^{-i\kappa r}\left(\frac{\pi}{2i\kappa r}\right)^{\frac{1}{2}}\left\{\frac{m'-m}{2m'} + \frac{\sigma'-\sigma}{\sigma'+\sigma}\cos\theta\right\}$$

$$= -\frac{2\pi\,.\,\pi c^2}{r^{\frac{1}{2}}\lambda^{\frac{3}{2}}}e^{-i(\kappa r + \frac{1}{4}\pi)}\left\{\frac{m'-m}{2m'} + \frac{\sigma'-\sigma}{\sigma'+\sigma}\cos\theta\right\} \text{...(14)}.$$

Hence, corresponding to the primary wave

$$\phi = \cos\frac{2\pi}{\lambda}(at + x) \text{....................(15)},$$

the scattered wave is approximately

$$\psi = -\frac{2\pi\,.\,\pi c^2}{r^{\frac{1}{2}}\lambda^{\frac{3}{2}}}\left\{\frac{m'-m}{2m'} + \frac{\sigma'-\sigma}{\sigma'+\sigma}\cos\theta\right\}\cos\frac{2\pi}{\lambda}(at - r - \tfrac{1}{8}\lambda) \text{......(16)}.$$

The fact that ψ varies inversely as $\lambda^{-\frac{3}{2}}$ might have been anticipated by the method of dimensions as in the corresponding problem for the sphere (§ 335). As in that case, the symmetrical part of the divergent wave depends upon the variation of compressibility, and would disappear in the application to an actual gas, and the term of the first order depends upon the variation of density.

By supposing σ' and m' to become infinite, in such a manner that their ratio remains finite, we obtain the solution corresponding to a rigid and immoveable obstacle,

$$\psi = -\frac{2\pi\,.\,\pi c^2}{r^{\frac{1}{2}}\lambda^{\frac{3}{2}}}(\tfrac{1}{2} + \cos\theta)\cos\frac{2\pi}{\lambda}(at - r - \tfrac{1}{8}\lambda) \text{ ...(17)}.$$

The analysis of this section is applicable to the mathematically analogous problem of finding the effect of a cylindrical obstacle

on plane waves of transverse vibration in an elastic solid, the direction of vibration being parallel to the axis of the cylinder. If the densities be σ, σ' and the rigidities be n, n', and γ denote the transverse displacement, the boundary conditions are

$$\gamma\,(\text{inside}) = \gamma\,(\text{outside}),$$

$$n'\frac{d\gamma}{dr}\,(\text{inside}) = n\frac{d\gamma}{dr}\,(\text{outside}).$$

The result is that, corresponding to the primary waves

$$\gamma = \cos\frac{2\pi}{\lambda}bt \quad\text{.....................(18)},$$

the disturbance is

$$\gamma = \frac{2\pi\,.\,\pi c^2}{\lambda^{\frac{3}{2}}r^{\frac{1}{2}}}\left\{\frac{\sigma'-\sigma}{2\sigma} - \frac{n'-n}{n'+n}\cos\theta\right\}\cos\frac{2\pi}{\lambda}\left(bt - r - \tfrac{1}{8}\lambda\right) \text{...(19)}.$$

For an application to the theory of light the reader is referred to a paper by the author, 'On the manufacture and theory of diffraction gratings[1].'

The exceeding smallness of the obstruction offered by fine wires or fibres to the passage of sound is strikingly illustrated in some of Tyndall's experiments. A piece of stiff felt half an inch in thickness allows much more sound to pass than a *wetted* pocket-handkerchief, which in consequence of the closing of its pores behaves rather as a thin lamina. For the same reason fogs, and even rain and snow, interfere but little with the free propagation of sounds of moderate wave-length. In the case of a hiss, or other very acute sound, the effect would perhaps be apparent.

[1] *Phil. Mag.* Vol. XLVII. 1874.

CHAPTER XIX.

FLUID FRICTION. PRINCIPLE OF DYNAMICAL SIMILARITY.

344. THE equations of Chapter XI. and the consequences that we have deduced from them are based upon the assumption (§ 236), that the mutual action between any two portions of fluid separated by an imaginary surface is normal to that surface. Actual fluids however do not come up to this ideal; in many phenomena the defect of fluidity, usually called viscosity or fluid friction, plays an important and even a preponderating part. It will therefore be proper to inquire whether the laws of aerial vibrations are sensibly influenced by the viscosity of air, and if so in what manner.

In order to understand clearly the nature of viscosity, let us conceive a fluid divided into parallel strata in such a manner that while each stratum moves in its own plane with uniform velocity, a change of velocity occurs in passing from one stratum to another. The simplest supposition which we can make is that the velocities of all the strata are in the same direction, but increase uniformly in magnitude as we pass along a line perpendicular to the planes of stratification. Under these circumstances a tangential force between contiguous strata is called into play, in the direction of the relative motion, and of magnitude proportional to the rate at which the velocity changes, and to a coefficient of viscosity, commonly denoted by the letter μ. Thus, if the strata be parallel to xy and the direction of their motion be parallel to y, the tangential force, reckoned (like a pressure) per unit of area, is

$$\mu \frac{dv}{dz} \qquad \text{..............................(1).}$$

The dimensions of μ are $[ML^{-1}T^{-1}]$.

The examination of the origin of the tangential force belongs to molecular science. It has been explained by Maxwell in ac-

cordance with the kinetic theory of gases as resulting from interchange of molecules between the strata, giving rise to diffusion of momentum. Both by theory and experiment the remarkable conclusion has been established that within wide limits the force is independent of the density of the gas. For air at θ° Centigrade Maxwell[1] found

$$\mu = \cdot 0001878\,(1 + \cdot 00366\,\theta) \quad\text{..................}(2),$$

the centimetre, gramme, and second being units.

345. The investigation of the equations of fluid motion in which regard is paid to viscous forces can scarcely be considered to belong to the subject of this work, but it may be of service to some readers to point out its close connection with the more generally known theory of solid elasticity.

The potential energy of unit of volume of uniformly strained isotropic matter may be expressed[2]

$$\begin{aligned} V &= \tfrac{1}{2} m\delta^2 + \tfrac{1}{2} n\,(e^2 + f^2 + g^2 - 2fg - 2ge - 2ef + a^2 + b^2 + c^2) \\ &= \tfrac{1}{2}\kappa\delta^2 + \tfrac{1}{2} n\,(2e^2 + 2f^2 + 2g^2 - \tfrac{2}{3}\delta^2 + a^2 + b^2 + c^2) \quad\text{........}(1), \end{aligned}$$

in which $\delta\,(=e+f+g)$ is the dilatation, e, f, g, a, b, c are the six components of strain, connected with the actual displacements α, β, γ by the equations

$$e = \frac{d\alpha}{dx}, \qquad f = \frac{d\beta}{dy}, \qquad g = \frac{d\gamma}{dz} \quad\text{..................}(2),$$

$$a = \frac{d\beta}{dz} + \frac{d\gamma}{dy}, \quad b = \frac{d\gamma}{dx} + \frac{d\alpha}{dz}, \quad c = \frac{d\alpha}{dy} + \frac{d\beta}{dx} \quad\text{............}(3),$$

and m, n, κ are constants of elasticity, connected by the equation

$$\kappa = m - \tfrac{1}{3} n \quad\text{..............................}(4),$$

of which n measures the *rigidity*, or resistance to *shearing*, and κ measures the resistance to change of *volume*. The components of stress P, Q, R, S, T, U, corresponding respectively to e, f, g, a, b, c, are found from V by simple differentiation with respect to those quantities; thus

$$P = \kappa\delta + 2n\,(e - \tfrac{1}{3}\delta) \ \ \&\text{c.} \quad\text{..................}(5),$$

$$S = na \ \ \&\text{c.} \quad\text{.................................}(6).$$

[1] On the Viscosity or Internal Friction of Air and other Gases. *Phil. Trans.* 1866.

[2] Thomson and Tait's *Natural Philosophy*. Appendix C.

If X, Y, Z be the components of the applied force reckoned per unit of volume, the equations of equilibrium are of the form

$$\frac{dP}{dx}+\frac{dU}{dy}+\frac{dT}{dz}+X=0 \text{ \&c.} \qquad (7),$$

from which the equations of motion are immediately obtainable by means of D'Alembert's principle. In terms of the displacements α, β, γ, these equations become

$$\kappa\frac{d\delta}{dx}+\tfrac{1}{3}n\frac{d\delta}{dx}+n\nabla^2\alpha+X=0 \text{ \&c.} \qquad (8),$$

where

$$\delta=\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz} \qquad (9).$$

In the ordinary theory of fluid friction no forces of restitution are included, but on the other hand, we have to consider viscous forces whose relation to the velocities (u, v, w) of the fluid elements is of precisely the same character as that of the forces of restitution to the displacements (α, β, γ) of an isotropic solid. Thus if δ' be the velocity of dilatation, so that

$$\delta'=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} \qquad (10),$$

the force parallel to x due to viscosity is, as in (8),

$$\kappa\frac{d\delta'}{dx}+\tfrac{1}{3}n\frac{d\delta'}{dx}+n\nabla^2u \qquad (11).$$

So far κ and n are arbitrary constants; but it has been argued with great force by Prof. Stokes, that there is no reason why a motion of dilatation uniform in all directions should give rise to viscous force, or cause the pressure to differ from the statical pressure corresponding to the actual density. In accordance with this argument we are to put $\kappa=0$; and, as appears from (6), n coincides with the quantity previously denoted by μ. The frictional terms are therefore

$$\mu\left\{\nabla^2u+\tfrac{1}{3}\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right\}, \text{ \&c.};$$

and (§ 237) the equations of motion take the form

$$\rho\left(\frac{Du}{Dt}-X\right)+\frac{dp}{dx}-\mu\nabla^2u-\tfrac{1}{3}\mu\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0 \qquad (12),$$

or, if there be no applied forces and the square of the motion be neglected,

$$\rho_0 \frac{du}{dt} + \frac{dp}{dx} - \mu \nabla^2 u - \tfrac{1}{3}\mu \frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0 \;\ldots\ldots(13).$$

We may observe that the dissipative forces here considered correspond to a dissipation function, whose form is the same with respect to u, v, w as that of V with respect to α, β, γ, in the theory of isotropic solids. Thus putting $\kappa = 0$, we have from (1)

$$F = \mu \iiint \left[2\left(\frac{du}{dx}\right)^2 + 2\left(\frac{dv}{dy}\right)^2 + 2\left(\frac{dw}{dz}\right)^2 - \tfrac{2}{3}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)^2 + \left(\frac{dv}{dz} + \frac{dw}{dy}\right)^2 + \left(\frac{dw}{dx} + \frac{du}{dz}\right)^2 + \left(\frac{du}{dy} + \frac{dv}{dx}\right)^2 \right] dx\,dy\,dz \ldots\ldots(14),$$

in agreement with Prof. Stokes' calculation[1]. The theory of friction for the case of a compressible fluid was first given by Poisson[2].

346. We will now apply the differential equations to the investigation of plane waves of sound. Supposing that v and w are zero and that u, p, &c. are functions of x only, we obtain from (13) § 345

$$\rho_0 \frac{du}{dt} + \frac{dp}{dx} - \frac{4\mu}{3}\frac{d^2u}{dx^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation of continuity (3) § 238 is in this case

$$\frac{ds}{dt} + \frac{du}{dx} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and the relation between the variable part of the pressure δp and the condensation s is as usual (§ 244)

$$\delta p = a^2 \rho_0 s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Thus, eliminating δp and s between (1), (2), (3), we obtain

$$\frac{d^2u}{dt^2} - a^2 \frac{d^2u}{dx^2} - \frac{4\mu}{3\rho_0}\frac{d^3u}{dx^2\,dt} = 0 \ldots\ldots\ldots\ldots\ldots(4),$$

which is the equation given by Stokes[3].

Let us now inquire how a train of harmonic waves of wave-length λ, which are maintained at the origin ($x = 0$), fade away

[1] *Cambridge Transactions*, 1851. § 49.
[2] *Journal de l'Ecole Polytechnique*, t. XIII. cah. 20, p. 139.
[3] *Cambridge Transactions*, 1845.

as x increases. Assuming that u varies as e^{int}, we find as in § 148,

$$y = Ae^{-\alpha x} \cos (nt - \beta x) \qquad (5),$$

where

$$\beta^2 - \alpha^2 = \frac{n^2 a^2}{a^4 + \dfrac{16\mu^2 n^2}{9\rho_0^2}}, \qquad 2\alpha\beta = \frac{\dfrac{4\mu n^3}{3\rho_0}}{a^4 + \dfrac{16\mu^2 n^2}{9\rho_0^2}} \qquad (6).$$

In the application to air at ordinary pressures μ may be considered to be a very small quantity and its square may be neglected. Thus

$$\beta = \frac{n}{a}, \qquad \alpha = \frac{2\mu n^2}{3\rho_0 a^3} \qquad (7).$$

It appears that to this order of approximation the velocity of sound is unaffected by fluid friction. If we replace n by $2\pi a\lambda^{-1}$, the expression for the coefficient of decay becomes

$$\alpha = \frac{8\pi^2\mu}{3\lambda^2\rho_0 a} \qquad (8),$$

shewing that the influence of viscosity is greatest on the waves of short wave-length. The amplitude is diminished in the ratio $e : 1$, when $x = \alpha^{-1}$. In C. G. S. measure we may take

$$\rho_0 = \cdot 0013, \quad \mu = \cdot 00019, \quad a = 33200;$$

whence

$$x = 8800\lambda^2 \qquad (9).$$

Thus the amplitude of waves of one centimetre wave-length is diminished in the ratio $e : 1$ after travelling a distance of 88 metres. A wave-length of 10 centimetres would correspond nearly to g^{iv}; for this case $x = 8800$ metres. It appears therefore that at atmospheric pressures the influence of friction is not likely to be sensible to ordinary observation, except near the upper limit of the musical scale. The mellowing of sounds by distance, as observed in mountainous countries, is perhaps to be attributed to friction, by the operation of which the higher and harsher components are gradually eliminated. It must often have been noticed that the sound s is scarcely, if at all, returned by echos, and I have found[1] that at a distance of 200 metres a powerful hiss loses its character, even when there is no reflection. Probably this effect also is due to viscosity.

[1] Acoustical Observations, *Phil. Mag.*, June, 1877.

In highly rarefied air the value of α as given in (8) is much increased, μ being constant. Sounds even of grave pitch may then be affected within moderate distances.

From the observations of Colladon in the lake of Geneva it would appear that in water grave sounds are more rapidly damped than acute sounds. At a moderate distance from a bell, struck under water, he found the sound short and sharp, without musical character.

347. The effect of viscosity in modifying the motion of air in contact with vibrating solids will be best understood from the solution of the problem for a very simple case given by Stokes. Let us suppose that an infinite plane (yz) executes harmonic vibrations in a direction (y) parallel to itself. The motion being in parallel strata, u and w vanish, and the variable quantities are functions of x only. The first of equations (13) § 345 shews that the pressure is constant; the corresponding equation in v takes the form

$$\frac{dv}{dt} = \frac{\mu}{\rho}\frac{d^2v}{dx^2} \quad\text{..............................(1)},$$

similar to the equation for the linear conduction of heat. If we now suppose that v is proportional to e^{int}, the resulting equation in x is

$$\frac{d^2v}{dx^2} = i\frac{n\rho}{\mu}v \quad\text{..........................(2)},$$

and its general solution

$$v = Ae^{-\gamma x} + Be^{+\gamma x} \quad\text{........................(3)},$$

where

$$\gamma = \sqrt{\left(\frac{n\rho}{2\mu}\right)}(1 \pm i) \quad\text{........................(4)}.$$

If the gas be on the positive side of the vibrating plane the motion is to vanish when $x = +\infty$. Hence $B = 0$, and the value of v becomes on rejection of the imaginary part

$$v = Ae^{-\sqrt{\left(\frac{n\rho}{2\mu}\right)}x}\cos\left\{nt - \sqrt{\left(\frac{n\rho}{2\mu}\right)}x\right\} \quad\text{..............(5)},$$

corresponding to the motion

$$V = A\cos nt \quad\text{...........................(6)}$$

at $x = 0$. The velocity of the fluid in contact with the plane is usually assumed to be the same as that of the plane itself on the

apparently sufficient ground that the contrary would imply an infinitely greater smoothness of the fluid with respect to the solid than with respect to itself. On this supposition (5) expresses the motion of the fluid on the positive side due to a motion of the plane given by (6).

The tangential force per unit area acting on the plane is

$$\mu \frac{dv}{dx}_{(x=0)},$$

or $\mu \sqrt{\left(\frac{n\rho}{2\mu}\right)}\{-\cos nt + \sin nt\} = -\sqrt{(\tfrac{1}{2}n\rho\mu)}\left(V + \frac{1}{n}\frac{dV}{dt}\right)$ ...(7),

if $A = 1$. The first term represents a dissipative force tending to stop the motion; the second represents a force equivalent to an increase in the inertia of the vibrating body. The magnitude of both forces depends upon the frequency of the vibration.

We will apply this result to calculate approximately the velocity of sound in tubes so narrow that the viscosity of air exercises a sensible influence. As in § 265, let X denote the total transfer of fluid across the section of the tube at the point x. The force, due to hydrostatic pressure, acting on the slice between x and $x + \delta x$ is, as usual,

$$-S\frac{dp}{dx}\delta x = a^2\rho\,\delta x\frac{d^2X}{dx^2} \text{ ....(8).}$$

The force due to viscosity may be inferred from the investigation for a vibrating plane, provided that the thickness of the layer of air adhering to the walls of the tube be small in comparison with the diameter. Thus, if P be the perimeter of the tube, and V be the velocity of the current at a distance from the walls of the tube, the tangential force on the slice, whose volume is $S\delta x$, is by (7)

$$-P\delta x\sqrt{(\tfrac{1}{2}n\rho\mu)}\left(V + \frac{1}{n}\frac{dV}{dt}\right),$$

or on replacing V by $\dfrac{dX}{dt} \div S$

$$-P\,\delta x\sqrt{(\tfrac{1}{2}n\rho\mu)}\left(\frac{dX}{dt} + \frac{1}{n}\frac{d^2X}{dt^2}\right) \div S \text{...............(9).}$$

The equation of motion *for this period* is therefore

$$\rho\delta x\frac{d^2X}{dt^2} + \sqrt{(\tfrac{1}{2}n\rho\mu)}\frac{P\delta x}{S}\left(\frac{dX}{dt} + \frac{1}{n}\frac{d^2X}{dt^2}\right) = a^2\rho\,\delta x\frac{d^2X}{dx^2},$$

or $$\frac{d^2X}{dt^2}\left\{1+\frac{P}{S}\sqrt{\left(\frac{\mu}{2n\rho}\right)}\right\}+\frac{P}{S}\sqrt{\left(\frac{n\mu}{\rho}\right)}\frac{dX}{dt}=a^2\frac{d^2X}{dx^2}\text{....(10).}$$

The velocity of sound is approximately

$$a\left\{1-\tfrac{1}{2}\frac{P}{S}\sqrt{\left(\frac{\mu}{2n\rho}\right)}\right\}\text{.....................(11),}$$

or in the case of a circular tube of radius r,

$$a\left\{1-\frac{1}{r}\sqrt{\left(\frac{\mu}{2n\rho}\right)}\right\}\text{.........................(12).}$$

The result expressed in (12) was first obtained by Helmholtz. An elaborate investigation of this problem has also been given by Kirchhoff[1], who included in his calculation not only the effect of friction but also that of the conduction of heat. Kirchhoff's result is of the same form as (12), but $\sqrt{(\mu\rho^{-1})}$ is replaced by the quantity (called γ)

$$\sqrt{\left(\frac{\mu}{\rho}\right)}+\left(\frac{a}{b}-\frac{b}{a}\right)\sqrt{\nu}\text{.......................(13),}$$

where b is Newton's value of the velocity of sound, and ν is a coefficient of conduction, equal according to the kinetic theory of gases to $\tfrac{5}{2}\mu\rho^{-1}$.

The diminution of the velocity of sound in narrow tubes, as indicated by the wave-length of stationary vibrations, was observed by Kundt (§ 260), and has been specially investigated by Schneebeli[2] and A. Seebeck[3]. It appears that the diminution of velocity varies as r^{-1}, in accordance with (12), but, when n varies, it is proportional rather to $n^{-\frac{3}{10}}$ than to $n^{-\frac{1}{2}}$. Since μ is independent of the density (ρ), the effect would be increased in rarefied air.

348. In the course of this work we have had frequent occasion to notice the importance of the conclusions that may be arrived at by the method of dimensions. Now that we are in a position to draw illustrations from a greater variety of acoustical phenomena relating to the vibrations of both solids and fluids, it will be convenient to resume the subject, and to develope somewhat in detail the principles upon which the method rests.

In the case of systems, such as bells or tuning-forks, formed of uniform isotropic material, and vibrating in virtue of elasticity, the

[1] *Pogg. Ann.* t. CXXXIV 177. 1868. [2] *Pogg. Ann.* t. CXXXVI. 296. 1869.
[3] *Pogg. Ann.* t. CXXXIX. 104. 1870.

acoustical elements are the shape, the linear dimension c, the constants of elasticity q and μ (§ 149), and the density ρ. Hence, by the method of dimensions, the periodic time varies *cœteris paribus* as the linear dimension, at least if the amplitude of vibration be in the same proportion; and, if the law of isochronism be assumed, the last-named restriction may be dispensed with. In fact, since the dimensions of q and ρ are respectively $[ML^{-1}T^{-2}]$ and $[ML^{-3}]$, while μ is a mere number, the only combination capable of representing a time is $q^{-\frac{1}{2}} . \rho^{\frac{1}{2}} . c$.

The argument which underlies this mathematical shorthand is of the following nature. Conceive two geometrically similar bodies, whose mechanical constitution at corresponding points is the same, to execute similar movements in such a manner that the corresponding changes occupy times[1] which are proportional to the linear dimensions—in the ratio, say, of $1 : n$. Then, if the one movement be possible as a consequence of the elastic forces, the other will be also. For the masses to be moved are as $1 : n^3$, the accelerations as $1 : n^{-1}$, and therefore the necessary forces are as $1 : n^2$; and, since the strains are the same, this is in fact the ratio of the elastic forces due to them when referred to corresponding areas. If the elastic forces are competent to produce the supposed motion in the first case, they are also competent to produce the supposed motion in the second case.

The dynamical similarity is disturbed by the operation of a force like gravity, proportional to the cubes, and not to the squares, of corresponding lines; but in cases where gravity is the sole motive power, dynamical similarity may be secured by a different relation between corresponding spaces and corresponding times. Thus if the ratio of corresponding spaces be $1 : n$, and that of corresponding times be $1 : n^{\frac{1}{2}}$, the accelerations are in both cases the same, and may be the effects of forces in the ratio $1 : n^3$ acting on masses which are in the same ratio. As examples coming under this head may be mentioned the common pendulum, sea-waves, whose velocity varies as the square root of the wave-length, and the whole theory of the comparison of ships and their models by which Mr Froude predicts the behaviour of ships from experiments made on models of moderate dimensions.

[1] The conception of an alteration of scale in space has been made familiar by the universal use of maps and models, but the corresponding conception for time is often less distinct. Reference to the case of a musical composition performed at different speeds may assist the imagination of the student.

The same comparison that we have employed above for elastic solids applies also to aerial vibrations. The pressures in the cases to be compared are the same, and therefore when acting over areas in the ratio $1 : n^2$, give forces in the same ratio. These forces operate on masses in the ratio $1 : n^3$, and therefore produce accelerations in the ratio $1 : n^{-1}$, which is the ratio of the actual accelerations when both spaces and times are as $1 : n$. Accordingly the periodic times of similar resonant cavities, filled with the same gas, are directly as the linear dimension—a very important law first formulated by Savart.

Since the same method of comparison applies both to elastic solids and to elastic fluids, an extension may be made to systems into which both kinds of vibration enter. For example, the scale of a system compounded of a tuning-fork and of an air resonator may be supposed to be altered without change in the motion other than that involved in taking the times in the same ratio as the linear dimensions.

Hitherto the alteration of scale has been supposed to be uniform in all dimensions, but there are cases, not coming under this head, to which the principle of dynamical similarity may be most usefully applied. Let us consider, for example, the flexural vibrations of a system composed of a thin elastic lamina, plane or curved. By §§ 214, 215 we see that the thickness of the lamina b, and the mechanical constants q and ρ, will occur only in the combinations qb^3 and $b\rho$, and thus a comparison may be made even although the alteration of thickness be not in the same proportion as for the other dimensions. If c be the linear dimension when the thickness is disregarded, the times must vary *cæteris paribus* as $q^{-\frac{1}{2}} . \rho^{\frac{1}{2}} . c^2 . b^{-1}$. For a given material, thickness, and shape, the times are therefore as the *squares* of the linear dimension. It must not be forgotten, however, that results such as these, which involve a law whose truth is only approximate, stand on a different level from the more immediate consequences of the principle of similarity.

THE END.

APPENDIX A. (§ 307).

The problem of determining the correction for the open end of a tube is one of considerable difficulty, even when there is an infinite flange. It is proved in the text (§ 307) that the correction α is greater than $\frac{1}{4}\pi R$, and less than $\frac{8}{3\pi}R$. The latter value is obtained by calculating the energy of the motion on the supposition that the velocity parallel to the axis is constant over the plane of the mouth, and comparing this energy with the square of the total current. The actual velocity, no doubt, increases from the centre outwards, becoming infinite at the sharp edge; and the assumption of a constant value is a somewhat violent one. Nevertheless the value of α so calculated turns out to be not greatly in excess of the truth. It is evident that we should be justified in expecting a very good result, if we assume an axial velocity of the form

$$1+\mu\frac{r^2}{R^2}+\mu'\frac{r^4}{R^4},$$

r denoting the distance of the point considered from the centre of the mouth, and then determine μ and μ' so as to make the whole energy a minimum. The energy so calculated, though necessarily in excess, must be a very good approximation to the truth.

In carrying out this plan we have two distinct problems to deal with, the determination of the motion (1) outside, and (2) inside the cylinder. The former, being the easier, we will take first.

The conditions are that ϕ vanish at infinity, and that when $x=0$, $\frac{d\phi}{dx}$ vanish, except over the area of the circle $r=R$, where

$$\frac{d\phi}{dx}=1+\mu\frac{r^2}{R^2}+\mu'\frac{r^4}{R^4}\qquad\qquad(1).$$

Under these circumstances we know (§ 278) that

$$\phi=-\frac{1}{2\pi}\iint\frac{d\phi}{dx}\frac{d\sigma}{\rho}\qquad\qquad(2),$$

where ρ denotes the distance of the point where ϕ is to be estimated from the element of area $d\sigma$. Now

$$2\,(\text{kinetic energy})^1 = -\tfrac{1}{2}\iint \phi \frac{d\phi}{dx} d\sigma = \frac{1}{2\pi}\iint \frac{d\phi}{dx} \cdot \iint \frac{d\phi}{dx}\frac{d\sigma}{\rho} \cdot d\sigma = \frac{P}{\pi},$$

if P represent the potential on itself of a disc of radius R, whose

$$\text{density} = 1 + \mu \frac{r^2}{R^2} + \mu' \frac{r^4}{R^4}.$$

The value of P is to be calculated by the method employed in the text (§ 307) for a uniform density. At the edge of the disc, when cut down to radius a, we have the potential

$$V = 4a + \frac{20}{9}\frac{\mu a^3}{R^2} + \frac{356}{225}\frac{\mu' a^5}{R^4} \quad\text{..................}(3),$$

and thus

$$P = \int_0^R 2\pi a\, da\, V\left\{1 + \mu \frac{a^2}{R^2} + \mu' \frac{a^4}{R^4}\right\}$$

$$= \frac{8\pi R^3}{3}\left\{1 + \frac{14}{15}\mu + \frac{5}{21}\mu^2 + \frac{314}{525}\mu' + \frac{214}{675}\mu\mu' + \frac{89}{825}\mu'^2\right\} \quad\text{.........}(4),$$

on effecting the integration. This quantity divided by π gives twice the kinetic energy of the motion defined by (1).

The total current

$$= \int_0^R 2\pi r\, dr\left(1 + \mu \frac{r^2}{R^2} + \mu' \frac{r^4}{R^4}\right) = \pi R^2\,(1 + \tfrac{1}{2}\mu + \tfrac{1}{3}\mu') \quad\text{.........}(5).$$

We have next to consider the problem of determining the motion of an incompressible fluid within a rigid cylinder under the conditions that the axial velocity shall be uniform when $x = -\infty$, and when $x = 0$ shall be of the form

$$\frac{d\phi}{dx} = 1 + \mu \frac{r^2}{R^2} + \mu' \frac{r^4}{R^4}.$$

It will conduce to clearness if we separate from ϕ, that part of it which corresponds to a uniform flow. Thus, if we take

$$\frac{d\phi}{dx} = 1 + \tfrac{1}{2}\mu + \tfrac{1}{3}\mu' + \frac{d\psi}{dx},$$

ψ will correspond to a motion, which vanishes when x is numerically great. When $x = 0$,

$$\frac{d\psi}{dx} = \mu\,(r^2 - \tfrac{1}{2}) + \mu'\,(r^4 - \tfrac{1}{3}) \quad\text{..................}(6),$$

if for the sake of brevity we put $R = 1$.

[1] The density of the fluid is supposed to be unity.

Now ψ may be expanded in the series

$$\psi = \Sigma a_p e^{px} J_0(pr) \qquad\qquad (7),$$

where p denotes a root of the equation

$$J_0'(p) = 0 \qquad\qquad (8)^1.$$

Each term of this series satisfies the condition of giving no radial velocity, when $r = 1$; and no motion of any kind, when $x = -\infty$. It remains to determine the coefficients a_p so as to satisfy (6), when $x = 0$. From $r = 0$ to $r = 1$, we must have

$$\Sigma p a_p J_0(pr) = \mu(r^2 - \tfrac{1}{2}) + \mu'(r^4 - \tfrac{1}{3}),$$

whence multiplying by $J_0(pr)\, r dr$ and integrating from 0 to 1,

$$p a_p [J_0(p)]^2 = 2\int_0^1 r dr J_0(pr)\{\mu(r^2 - \tfrac{1}{2}) + \mu'(r^4 - \tfrac{1}{3})\},$$

every term on the left, except one, vanishing by the property of the functions. For the right-hand side we have

$$\int_0^1 r dr J_0(pr) = 0,$$

$$\int_0^1 r^3 dr J_0(pr) = \frac{2}{p^2} J_0(p),$$

$$\int_0^1 r^5 dr J_0(pr) = \left(\frac{4}{p^2} - \frac{32}{p^4}\right) J_0(p);$$

so that

$$a_p = \frac{4}{p^3 J_0(p)}\left\{\mu + 2\mu'\left(1 - \frac{8}{p^2}\right)\right\} \qquad\qquad (9).$$

The velocity-potential ϕ of the whole motion is thus

$$\phi = (1 + \tfrac{1}{2}\mu + \tfrac{1}{3}\mu')x + 4\Sigma \frac{\mu + 2\mu'(1 - 8p^{-2})}{p^3 J_0(p)} e^{px} J_0(pr) \ldots\ldots (10),$$

the summation extending to all the admissible values of p. We have now to find the energy of motion of so much of the fluid as is included between $x = 0$, and $x = -l$, where l is so great that the velocity is there sensibly constant.

By Green's theorem

$$2\,(\text{kinetic energy}) = \int_0^1 \phi \frac{d\phi}{dx} 2\pi r\, dr \quad (x = 0) - \int_0^1 \phi \frac{d\phi}{dx} 2\pi r\, dr \quad (x = -l).$$

1 The numerical values of the roots are approximately

$p_1 = 3{\cdot}831705$, $p_2 = 7{\cdot}015$, $p_3 = 10{\cdot}174$,
$p_4 = 13{\cdot}324$, $p_5 = 16{\cdot}471$, $p_6 = 19{\cdot}616$.

Now, when $x = -l$,

$$\phi = -(1 + \tfrac{1}{2}\mu + \tfrac{1}{3}\mu')\, l,$$

$$\frac{d\phi}{dx} = 1 + \tfrac{1}{2}\mu + \tfrac{1}{3}\mu';$$

so that the second term is $\pi l(1 + \tfrac{1}{2}\mu + \tfrac{1}{3}\mu')^2$.

In calculating the first term, we must remember that if p_1 and p_2 be two different values of p,

$$\int_0^1 2\pi r\, dr\, J_0(p_1 r)\, J_0(p_2 r) = 0.$$

Thus

$$\int_0^1 2\pi r\, dr\, \phi \frac{d\phi}{dx}(x=0) = 16\, \Sigma\, \frac{\{\mu + 2\mu'(1 - 8p^{-2})\}^2}{p^4 [J_0(p)]^2} \int_0^1 2\pi r\, dr\, [J_0(pr)]^2$$

$$= 16\pi\, \Sigma \left\{\mu + 2\mu'\left(1 - \frac{8}{p^2}\right)\right\}^2 p^{-5}.$$

Accordingly, on restoring R,

$$2\,(\text{kinetic energy}) = \pi R^2 l\, (1 + \tfrac{1}{2}\mu + \tfrac{1}{3}\mu')^2$$
$$+ 16\pi R^3\, \Sigma \left\{\mu + 2\mu'\left(1 - \frac{8}{p^2}\right)\right\}^2 p^{-5}.$$

To this must be added the energy of the motion on the positive side of $x = 0$. On the whole

$$\frac{2\ \text{kinetic energy}}{(\text{current})^2} = \frac{l}{\pi R^2} + \frac{16}{\pi R\,(1 + \frac{1}{2}\mu + \frac{1}{3}\mu')^2}\, \Sigma \left\{\mu + 2\mu'\left(1 - \frac{8}{p^2}\right)\right\}^2 p^{-5}$$
$$+ \frac{8}{3\pi^2 R}\, \frac{1 + \frac{14}{15}\mu + \frac{5}{21}\mu^2 + \frac{314}{525}\mu' + \frac{214}{675}\mu\mu' + \frac{89}{825}\mu'^2}{(1 + \frac{1}{2}\mu + \frac{1}{3}\mu')^2}$$

Hence, if α be the correction to the length,

$$\frac{3\pi\alpha}{8R} = [1 + \tfrac{14}{15}\mu + \tfrac{314}{525}\mu' + (6\pi\, \Sigma p^{-5} + \tfrac{5}{21})\,\mu^2$$
$$+ \{24\pi\,(\Sigma p^{-5} - 8\Sigma p^{-7}) + \tfrac{214}{675}\}\,\mu\mu'$$
$$+ \{24\pi\,(\Sigma p^{-5} - 16\Sigma p^{-7} + 64\Sigma p^{-9}) + \tfrac{89}{825}\}\,\mu'^2] \div (1 + \tfrac{1}{2}\mu + \tfrac{1}{3}\mu')^2.$$

By numerical calculation from the values of p

$$\Sigma p^{-5} = \cdot 00128266; \quad \Sigma p^{-5} - 8\Sigma p^{-7} = \cdot 00061255,$$
$$\Sigma p^{-5} - 16\Sigma p^{-7} + 64\Sigma p^{-9} = \cdot 00030351,$$

and thus
$$\frac{3\pi\alpha}{8R} = [1 + \cdot 9333333\,\mu + \cdot 5980951\,\mu'$$
$$+ \cdot 2622728\,\mu^2 + \cdot 363223\,\mu\mu' + \cdot 1307634\,\mu'^2] \div (1 + \tfrac{1}{2}\mu + \tfrac{1}{3}\mu')^2$$
$$= 1 - \frac{\cdot 0666667\,\mu + \cdot 0685716\,\mu' - \cdot 0122728\,\mu^2 - \cdot 029890\,\mu\mu' - \cdot 0196523\,\mu'^2}{(1 + \frac{1}{2}\mu + \frac{1}{3}\mu')^2} \qquad \text{.....................(11).}$$

The fraction on the right is the ratio of two quadratic functions of

μ, μ', and our object is to determine its maximum value. In general if S and S' be two quadratic functions, the maximum and minimum values of $z = S \div S'$ are given by the cubic equation

$$-\Delta z^{-3} + \Theta z^{-2} - \Theta' z^{-1} + \Delta' = 0,$$

where
$$S = a\mu^2 + b\mu'^2 + c + 2f\mu' + 2g\mu + 2h\mu\mu',$$
$$S' = a'\mu^2 + b'\mu'^2 + c' + 2f'\mu' + 2g'\mu + 2h'\mu\mu',$$
$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = \frac{(h^2 - ab)(g^2 - ac) - (hg - af)^2}{a},$$
$$\Theta = (bc - f^2)a' + (ca - g^2)b' + (ab - h^2)c'$$
$$+ 2(gh - af)f' + 2(hf - bg)g' + 2(fg - ch)h',$$

and Θ', Δ', are derived from Θ and Δ by interchanging the accented and unaccented letters.

In the present case, since S' is a product of linear factors, $\Delta' = 0$; and since the two factors are the same, $\Theta' = 0$, so that $z = \Delta \div \Theta$ simply. Substituting the numerical values, and effecting the calculations, we find $z = \cdot 0289861$, which is the maximum value of the fraction consistent with real values of μ and μ'.

The corresponding value of a is $\cdot 82122R$, than which the true correction cannot be greater.

If we assume $\mu' = 0$, the greatest value of z then possible is $\cdot 021363$, which gives

$$a = \cdot 828146R.$$

On the other hand if we put $\mu = 0$, the maximum value of z comes out $\cdot 027653$, whence

$$a = \cdot 825353R.$$

It would appear from this result that the variable part of the normal velocity at the mouth is better represented by a term varying as r^4, than by one varying as r^2.

The value $a = \cdot 8242R$ is probably pretty close to the truth. If the normal velocity be assumed constant, $a = \cdot 848826R$; if of the form $1 + \mu r^2$, $a = \cdot 82815R$, when μ is suitably determined; and when the form $1 + \mu r^2 + \mu' r^4$, containing another arbitrary constant, is made the foundation of the calculation, we get $a = \cdot 8242R$.

The true value of a is probably about $\cdot 82R$.

In the case of $\mu = 0$, the minimum energy corresponds to $\mu' = 1 \cdot 103$, so that

$$\frac{d\phi}{dx} = 1 + 1 \cdot 103\frac{r^4}{R^4}.$$

On this supposition the normal velocity of the edge ($r = R$) would be about double of that near the centre.

[1] Notes on Bessel's functions. *Phil. Mag.* Nov. 1872.

NOTE TO § 273.

A method of obtaining Poisson's solution (8) given by Liouville[1] is worthy of notice.

If r be the polar radius vector measured from any point O, and the general differential equation be integrated over the volume included between spherical surfaces of radii r and $r+dr$, we find on transformation of the second integral by Green's theorem

$$\frac{d^2(r\lambda)}{dt^2}=a^2\frac{d^2(r\lambda)}{dr^2}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\alpha),$$

in which $\lambda=\iint\phi\,d\sigma$, that is to say is proportional to the mean value of ϕ reckoned over the spherical surface of radius r. Equation (α) may be regarded as an extension of (1) § 279; it may also be proved from the expression (5) § 241 for $\nabla^2\phi$ in terms of the ordinary polar co-ordinates r, θ, ω.

The general solution of (α) is

$$r\lambda=\chi(at+r)+\theta(at-r)\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta),$$

where χ and θ are arbitrary functions; but, as in § 279, if the pole be not a source, $\chi(at)+\theta(at)=0$, so that

$$r\lambda=\chi(at+r)-\chi(at-r)\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\gamma).$$

It appears from (γ) that at O, when $r=0$, $\lambda=2\chi'(at)$, which is therefore also the value of $4\pi\phi$ at O at time t. Again from (γ)

$$2\chi'(at+r)=\frac{d(r\lambda)}{d(at)}+\frac{d(r\lambda)}{dr},$$

so that

$$2\chi'(r)=\left[\frac{d(r\lambda)}{d(at)}+\frac{d(r\lambda)}{dr}\right]_{(t=0)},$$

or in the notation of § 273

$$2\chi'(r)=\frac{r}{a}\iint F(r)\,d\sigma+\frac{d}{dr}\left[r\iint f(r)\,d\sigma\right]\;\ldots\ldots\ldots\ldots(\delta).$$

By writing at in place of r in (δ) we obtain the value of $2\chi'(at)$, or $4\pi\phi$, which agrees with (8) § 273.

[1] Liouville, tom. I. p. 1, 1856.

Note on Progressive Waves. From the Proceedings of the London Mathematical Society, Vol. IX. No. 125.

It has often been remarked that, when a group of waves advances into still water, the velocity of the group is less than that of the individual waves of which it is composed; the waves appear to advance through the group, dying away as they approach its anterior limit. This phenomenon was, I believe, first explained by Stokes, who regarded the group as formed by the superposition of two infinite trains of waves, of equal amplitudes and of nearly equal wave-lengths, advancing in the same direction. My attention was called to the subject about two years since by Mr Froude, and the same explanation then occurred to me independently[1]. In my book on the "Theory of Sound" (§ 191), I have considered the question more generally, and have shown that, if V be the velocity of propagation of any kind of waves whose wave-length is λ, and $\kappa = 2\pi\lambda^{-1}$, then U, the velocity of a group composed of a great number of waves, and moving into an undisturbed part of the medium, is expressed by

$$U = \frac{d(\kappa V)}{d\kappa} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or, as we may also write it,

$$U : V = 1 + \frac{d \log V}{d \log \kappa} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Thus, if $V \propto \lambda^n$, $\qquad U = (1 - n) V \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$

In fact, if the two infinite trains be represented by $\cos \kappa (Vt - x)$ and $\cos \kappa' (V't - x)$, their resultant is represented by

$$\cos \kappa (Vt - x) + \cos \kappa' (V't - x),$$

[1] Another phenomenon, also mentioned to me by Mr Froude, admits of a similar explanation. A steam launch moving quickly through the water is accompanied by a peculiar system of diverging waves, of which the most striking feature is the obliquity of the line containing the greatest elevations of successive waves to the wave-fronts. This wave pattern may be explained by the superposition of two (or more) infinite trains of waves, of slightly differing wave-lengths, whose directions and velocities of propagation are so related in each case that there is no change of position relatively to the boat. The mode of composition will be best understood by drawing on paper two sets of parallel and equidistant lines, subject to the above condition, to represent the crests of the component trains. In the case of two trains of slightly different wave-lengths, it may be proved that the tangent of the angle between the line of maxima and the wave-fronts is half the tangent of the angle between the wave-fronts and the boat's course.

which is equal to

$$2\cos\left\{\frac{\kappa'V'-\kappa V}{2}t-\frac{\kappa'-\kappa}{2}x\right\}.\cos\left\{\frac{\kappa'V'+\kappa V}{2}t-\frac{\kappa'+\kappa}{2}x\right\}.$$

If $\kappa'-\kappa$, $V'-V$ be small, we have a train of waves whose amplitude varies slowly from one point to another between the limits 0 and 2, forming a series of groups separated from one another by regions comparatively free from disturbance. The position at time t of the middle of that group, which was initially at the origin, is given by

$$(\kappa'V'-\kappa V)\,t-(\kappa'-\kappa)\,x=0,$$

which shews that the velocity of the group is $(\kappa'V'-\kappa V)\div(\kappa'-\kappa)$. In the limit, when the number of waves in each group is indefinitely great, this result coincides with (1).

The following particular cases are worth notice, and are here tabulated for convenience of comparison :—

$V\propto\lambda$,	$U=0$,	Reynolds' disconnected pendulums.
$V\propto\lambda^{\frac{1}{2}}$,	$U=\frac{1}{2}V$,	Deep-water gravity waves.
$V\propto\lambda^{0}$,	$U=V$,	Aërial waves, &c.
$V\propto\lambda^{-\frac{1}{2}}$,	$U=\frac{3}{2}V$,	Capillary water waves.
$V\propto\lambda^{-1}$,	$U=2V$,	Flexural waves.

The capillary water waves are those whose wave-length is so small that the force of restitution due to capillarity largely exceeds that due to gravity. Their theory has been given by Thomson (*Phil. Mag.*, Nov. 1871). The flexural waves, for which $U=2V$, are those corresponding to the bending of an elastic rod or plate ("Theory of Sound," § 191).

In a paper read at the Plymouth meeting of the British Association (afterwards printed in "Nature," Aug. 23, 1877), Prof. Osborne Reynolds gave a dynamical explanation of the fact that a group of deep-water waves advances with only half the rapidity of the individual waves. It appears that the energy propagated across any point, when a train of waves is passing, is only one-half of the energy necessary to supply the waves which pass in the same time, so that, if the train of waves be limited, it is impossible that its front can be propagated with the full velocity of the waves, because this would imply the acquisition of more energy than can in fact be supplied. Prof. Reynolds did not contemplate the cases where *more* energy is propagated than corresponds to the waves passing in the same time; but his argument, applied conversely to the results already given, shews that such cases must exist. The ratio of the energy propagated to that of the passing waves is $U : V$; thus the energy propagated in the unit time is $U : V$

of that existing in a length V, or U times that existing in the unit length. Accordingly

Energy propagated in unit time : Energy contained (on an average) in unit length $= d(\kappa V) : d\kappa$, by (1).

As an example, I will take the case of small irrotational waves in water of finite depth l[1]. If z be measured downwards from the surface, and the elevation (h) of the wave be denoted by

$$h = H\cos(nt - \kappa x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

in which $n = \kappa V$, the corresponding velocity-potential (ϕ) is

$$\phi = -VH\frac{e^{\kappa(z-l)} + e^{-\kappa(z-l)}}{e^{\kappa l} - e^{-\kappa l}}\sin(nt - \kappa x) \quad \ldots\ldots\ldots\ldots(5).$$

This value of ϕ satisfies the general differential equation for irrotational motion ($\nabla^2\phi = 0$), makes the vertical velocity $\frac{d\phi}{dz}$ zero when $z = l$, and $= -\frac{dh}{dt}$ when $z = 0$. The velocity of propagation is given by

$$V^2 = \frac{g}{\kappa}\frac{e^{\kappa l} - e^{-\kappa l}}{e^{\kappa l} + e^{-\kappa l}} \quad \ldots\ldots(6).$$

We may now calculate the energy contained in a length x, which is supposed to include so great a number of waves that fractional parts may be left out of account.

For the potential energy we have

$$V_1 = g\rho\iint_0^h z\,dz\,dx = \tfrac{1}{2}g\rho\int h^2\,dx = \tfrac{1}{4}g\rho H^2 . x \quad \ldots\ldots\ldots(7).$$

For the kinetic energy,

$$T = \tfrac{1}{2}\rho\iint\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\}dx\,dz$$

$$= \tfrac{1}{2}\rho\int\left(\phi\frac{d\phi}{dz}\right)_{z=0}dx = \tfrac{1}{4}g\rho H^2 . x \quad \ldots\ldots\ldots\ldots(8),$$

by (1) and (6). If, in accordance with the argument advanced at the end of this paper, the equality of V_1 and T be assumed, the value of the velocity of propagation follows from the present expressions. The whole energy in the waves occupying a length x is therefore (for each unit of breadth) $V_1 + T = \tfrac{1}{2}g\rho H^2 . x \ldots\ldots\ldots\ldots\ldots(9)$,

H denoting the maximum elevation.

[1] Prof. Reynolds considers the trochoidal wave of Rankine and Froude, which involves molecular rotation.

We have next to calculate the energy propagated in time t across a plane for which x is constant, or, in other words, the work (W) that must be done in order to sustain the motion of the plane (considered as a flexible lamina) in the face of the fluid pressures acting upon the front of it. The variable part of the pressure (δp), at depth z, is given by

$$\delta p = -\rho\frac{d\phi}{dt} = -nVH\frac{e^{\kappa(z-l)}+e^{-\kappa(z-l)}}{e^{\kappa l}-e^{-\kappa l}}\cos(nt-\kappa x),$$

while for the horizontal velocity

$$\frac{d\phi}{dx} = \kappa VH\frac{e^{\kappa(z-l)}+e^{-\kappa(z-l)}}{e^{\kappa l}-e^{-\kappa l}}\cos(nt-\kappa x);$$

so that $W = \iint \delta p\frac{d\phi}{dx}\,dz\,dt - \frac{1}{4}g\rho H^2 . Vt . \left[1+\frac{4\kappa l}{e^{2\kappa l}-e^{-2\kappa l}}\right]$. .. .. (10),

on integration. From the value of V in (6) it may be proved that

$$\frac{d(\kappa V)}{d\kappa} = \frac{1}{2}V\left\{1+\frac{1}{V^2}\frac{d(\kappa V^2)}{d\kappa}\right\} = \frac{1}{2}V\left\{1+\frac{4\kappa l}{e^{2\kappa l}-e^{-2\kappa l}}\right\};$$

and it is thus verified that the value of W for a unit time

$$= \frac{d(\kappa V)}{d\kappa}\times \text{energy in unit length}.$$

As an example of the direct calculation of U, we may take the case of waves moving under the joint influence of gravity and cohesion.

It is proved by Thomson that

$$V^2 = \frac{g}{\kappa}+T'\kappa \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11),$$

where T' is the cohesive tension. Hence

$$U = \frac{1}{2}V\left\{1+\frac{1}{V^2}\frac{d(\kappa V^2)}{d\kappa}\right\} = \frac{1}{2}V\frac{g+3\kappa^2T'}{g+\kappa^2T'} \quad\ldots\ldots\ldots\ldots (12).$$

When κ is small, the surface tension is negligible, and then $U=\frac{1}{2}V$; but when, on the contrary, κ is large, $U=\frac{3}{2}V$, as has already been stated. When $T'\kappa^2=g$, $U=V$. This corresponds to the minimum velocity of propagation investigated by Thomson.

Although the argument from interference groups seems satisfactory, an independent investigation is desirable of the relation between energy existing and energy propagated. For some time I was at a loss for a method applicable to all kinds of waves, not seeing in particular why the comparison of energies should introduce the consideration of

a variation of wave-length. The following investigation, in which the increment of wave-length is *imaginary*, may perhaps be considered to meet the want:—

Let us suppose that the motion of every part of the medium is resisted by a force of very small magnitude proportional to the mass and to the velocity of the part, the effect of which will be that waves generated at the origin gradually die away as x increases. The motion, which in the absence of friction would be represented by $\cos(nt-\kappa x)$, under the influence of friction is represented by $e^{-\mu x}\cos(nt-\kappa x)$, where μ is a small positive coefficient. In strictness the value of κ is also altered by the friction; but the alteration is of the second order as regards the frictional forces, and may be omitted under the circumstances here supposed. The energy of the waves per unit length at any stage of degradation is proportional to the square of the amplitude, and thus the whole energy on the positive side of the origin is to the energy of so much of the waves at their greatest value, i.e., at the origin, as would be contained in the unit of length, as $\int_0^\infty e^{-2\mu x}dx : 1$, or as $(2\mu)^{-1} : 1$. The energy transmitted through the origin in the unit time is the same as the energy dissipated; and, if the frictional force acting on the element of mass m be hmv, where v is the velocity of the element and h is constant, the energy dissipated in unit time is $h\Sigma mv^2$ or $2hT$, T being the kinetic energy. Thus, on the assumption that the kinetic energy is half the whole energy, we find that the energy transmitted in the unit time is to the greatest energy existing in the unit length as $u : 2\mu$. It remains to find the connection between h and μ.

For this purpose it will be convenient to regard $\cos(nt-\kappa x)$ as the real part of $e^{int}e^{-i\kappa x}$, and to inquire how κ is affected, when n is given, by the introduction of friction. Now the effect of friction is represented in the differential equations of motion by the substitution of $\frac{d^2}{dt^2}+h\frac{d}{dt}$ in place of $\frac{d^2}{dt^2}$, or, since the whole motion is proportional to e^{int}, by substituting $-n^2+ihn$ for $-n^2$. Hence the introduction of friction corresponds to an alteration of n from n to $n-\frac{1}{2}ih$ (the square of h being neglected); and accordingly κ is altered from κ to $\kappa-\frac{1}{2}ih\frac{d\kappa}{dn}$. The solution thus becomes $e^{-\frac{1}{2}h\frac{d\kappa}{dn}x}e^{i(nt-\kappa x)}$, or, when the imaginary part is rejected, $e^{-\frac{1}{2}h\frac{d\kappa}{dn}x}\cos(nt-\kappa x)$; so that $\mu=\frac{1}{2}h\frac{d\kappa}{dn}$, and $h : 2\mu=\frac{dn}{d\kappa}$. The ratio of the energy transmitted in the unit time to

the energy existing in the unit length is therefore expressed by $\frac{dn}{d\kappa}$ or $\frac{d(\kappa V)}{d\kappa}$, as was to be proved.

It has often been noticed, in particular cases of progressive waves, that the potential and kinetic energies are equal; but I do not call to mind any general treatment of the question. The theorem is not usually true for the individual parts of the medium[1], but must be understood to refer either to an integral number of wave-lengths, or to a space so considerable that the outstanding fractional parts of waves may be left out of account. As an example well adapted to give insight into the question, I will take the case of a uniform stretched circular membrane ("Theory of Sound," § 200) vibrating with a given number of nodal circles and diameters. The fundamental modes are not quite determinate in consequence of the symmetry, for any diameter may be made nodal. In order to get rid of this indeterminateness, we may suppose the membrane to carry a small load attached to it anywhere except on a nodal circle. There are then two definite fundamental modes, in one of which the load lies on a nodal diameter, thus producing no effect, and in the other midway between nodal diameters, where it produces a maximum effect ("Theory of Sound," § 208). If vibrations of both modes are going on simultaneously, the potential and kinetic energies of the whole motion may be calculated by *simple addition* of those of the components. Let us now, supposing the load to diminish without limit, imagine that the vibrations are of equal amplitude and differ in phase by a quarter of a period. The result is a *progressive* wave, whose potential and kinetic energies are the sums of those of the stationary waves of which it is composed. For the first component we have $V_1 = E\cos^2 nt$, $T_1 = E\sin^2 nt$; and for the second component, $V_2 = E\sin^2 nt$, $T_2 = E\cos^2 nt$; so that $V_1 + V_2 = T_1 + T_2 = E$, or the potential and kinetic energies of the progressive wave are equal, being the same as the whole energy of either of the components. The method of proof here employed appears to be sufficiently general, though it is rather difficult to express it in language which is appropriate to all kinds of waves.

[1] Aërial waves are an important exception.

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